STOCHASTIC PRODUCTION AND INVENTORY MODELS WITH LIMITED RESOURCES

By

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This dissertation considers issues regarding stochastic systems with limited resources. We study stochastic multi-item inventory models with limited warehouse capacity for cases of equal and unequal replenishment intervals. We also discuss stochastic production systems in which a single machine is used for a series of customer orders.

We first characterize a stochastic multi-item inventory model for the case of equal replenishment intervals with limited warehouse capacity by using dynamic programming. We use induction to derive the convexity of objective cost function and show that a myopic replenishment policy is optimal for both finite and infinite planning horizons.

For the unequal replenishment interval case, we completely characterized the optimal replenishment policy for the two-item, two-period case. For more general cases, because of the inability to show convexity of the objective function, we construct three heuristics to determine replenishment quantities for the case of unequal replenishment intervals. Numerical testing of these heuristics suggests that they yield near optimal solutions for a small set of problems. The performance of these heuristics was evaluated for a number of problems with various cost and demand parameters.

For stochastic production systems with limited resources, we have modeled the problem of setting optimal planned order leadtime in a single stage, make-to-order production system. As a solution approach, we derived the distribution of actual process completion time of individual orders and then determined the optimal planned leadtime.

CHAPTER 1
INTRODUCTION

Supply chain management has become an important principle as industries strive for operational excellence to optimize resources and to reduce costs. By running an effective supply chain, companies can meet their customer demand satisfactorily and improve their competitive position in markets. One of the tools that lends itself to improving SCM is outsourcing. That is, you must determine when to have someone else do work for you such as warehousing, product shipment, or manufacturing components, so that you can concentrate on your *core competencies* –those things that you do best. For example, Carrier's McMinville manufacturing plant (Swenson 1998) developed a partnership with a supplier of non-production service parts. This supplier staffs McMinville's service parts warehouse and is fully responsible for purchasing stocking, and scheduling service parts. The supplier can source service parts and actually run the warehouse more cost- effectively than McMinville can with its own employees. Federal Express's warehousing and distribution service for several manufacturers (such as Dell Computer, Hewlett-Packard and National Semiconductor) is another outstanding example of outsourcing. Where once the transportation process was seen as a lost cause, it is now viewed as an important part of supply chain where business is focusing on reducing the cost of goods. When coupled with warehousing, the distribution process commands respect. Federal Express is currently responsible for managing the aforementioned companies' products warehouse

as well as the entire distribution process from those manufacturers to their respective customers. Continuous improvement in reducing average delivery time by companies such as Federal Express and UPS becomes a critical piece of cost reduction to those manufacturers.

On the other hand, on the service-providers' side, for example, Federal Express runs a giant warehouse to accommodate those products from outsourcing-partnerships with manufacturers as well as, of course, its primary business such as mail and parcels. Traditionally, warehouses function mainly to store products supplied from manufacturers or suppliers. Manufacturers decide when, for how long, and how many products are stored. Warehouses themselves have no control over supply, but mainly concentrate on warehouse or storage management. Today, instead of functioning merely as service providers of products warehousing and distribution, these companies could play a central role in a supply chain in which they are involved. In other words, once the service providers develop a strong relationship with their partners, they might expand their roles not only to play as current service-providers but also to control, at least partially, manufacturers' production schedules. This could be done by offering counter-compensation to manufacturers through a negotiation process, although an interesting issue - how to negotiate - is not covered in this study. For example, Federal Express might offer a discounted transportation cost to Dell Computer as compensation. Once these service providers are in the driver's seat in their supply chain, they can contact their partners' customers directly and collect information about demand. Based on this customer information, service providers can ask their partners to produce a certain amount of product to

meet future demand.

Once in a position to play a central role in their total supply chain, and as the number of partnerships grow, service providers confront how to run the entire system effectively with their own limited resources, such as warehouses and transportation resources. For example, Federal Express might need a well-defined strategy to accommodate products from its partners effectively and to meet their customer demands, given that its warehouse has a limited capacity (see Figure 1.1). In other words, Federal Express may need to determine optimal replenishment policies for minimizing costs. Meanwhile, the company maintains partnerships with several different manufacturers at the same time and each of them likely has a distinct production schedule. The company's warehouse thus would need a policy of reviewing its inventory irregularly or non-periodically instead of running a conventional periodic-review policy.

When individual manufacturers have the same replenishment interval for products, and thus the system has a periodic-review inventory policy, deriving an optimal replenishment policy would not be difficult (Chapter 2). However, when each manufacturer has different replenishment schedules or has unequal replenishment intervals, the periodic-review policy is not valid anymore. It would be difficult to determine an optimal replenishment policy. In this study, we focus on unequal replenishment intervals (either different replenishment schedules or unequal lengths of replenishment intervals). We try to determine the optimal replenishment policy for minimizing warehouse costs.

Stochastic production and inventory models involving deterministic, capac-

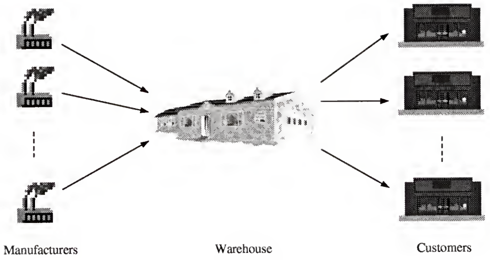


Figure 1.1: A multi-product inventory system.

ity (production) constrained periodic-review policies have attracted the attention of many researchers. Evans (1967) first considers this issue, in which he models periodic-review production and inventory systems with multiple products, random demands and a finite planning horizon. He develops the form of the optimal policy for the multi-product control appropriate for the system. Since then, much of the literature has studied periodic-review, single-product systems with production capacity constraint. Florian and Klein (1971) and De Kok et al. (1984) characterize the structure of the optimal solution to a multi-period, single-item production model with a capacity constraint. Federgruen and Zipkin (1986a,b) show that a modified base-stock policy is optimal under the discounted cost-criterion and the average cost-criterion, respectively, both for finite and infinite planning horizons. The modified base-stock policy is such that when initial stock is below a certain critical number, produce enough to bring total stock up to that number, or as close to it as possible,

given the limited capacity; otherwise, do not produce. They also characterize the optimal base-stock levels in several ways. Kapuscinski and Tayur (1998) provide simpler proof of optimality than Federgruen and Zipkin (1986a) for the infinite-horizon discounted cost case, based on results from Bertsekas (19988). Ciarallo et al. (1994) and Wang and Gerchak (1996) analyze a production model with variable capacity in a similar environment to that described by Federgruen and Zipkin (1986a). Wang and Gerchak (1996) also incorporate variable capacity explicitly into continuous review models.

DeCroix and Arreola-Risa (1998) study an infinite-horizon version of the multi-product case. They establish the optimal policy for the case of homogeneous products, and propose a heuristic policy for heterogeneous products by generalizing the optimal policy for the homogeneous product case. Glasserman (1996) addresses a similar problem to DeCroix and Arreola-Risa (1998) in a continuous-review system. He presents a procedure for choosing base-stock levels and capacity allocations that are asymptotically optimal, but assumes that a fixed proportion of total capacity is dedicated exclusively to each product. The use of asymptotic analysis is similar in the spirit to Anantharam (1989). His static allocation problem contrasts with the dynamic scheduling problem addressed in Wein (1989) and Zheng and Zipkin (1990) and the priority scheme in Carr et al. (1996). Lau and Lau (1988, 1996) present formulations and solution procedures for handling a multi-product newsboy problem under one resource and multiple resource constraint. Nahmias and Schmidt (1984) also investigate several heuristics for a single-period, multi-item inventory problem with a resource constraint.

A multi-product dynamic nonstationary inventory problem with limited warehouse capacity is first considered in Veinott (1965). He provides conditions that ensure that the base stock ordering policy is optimal in a periodic-review inventory system with a finite horizon. Ignall and Veinott (1969) show in the stationary case that a myopic ordering policy is optimal for a sequence of periods for all initial inventory levels. Beyer et al. (2000a,b), use the dynamic programming approach to derive the optimality of the ordering policy for the average cost problem and to prove the convexity of the objective cost function as we did in this study coincidentally. Beyer et al. extensively also prove the optimality of the modified base-stock policy in the discounted cost version of the problem. In Federgruen and Zipkin (1986a), they consider both production and inventory capacities with single-product systems, and derive the convexity of the expected cost function.

Anily (1991) and Gallego et al. (1996) study a multi-item replenishment problem with deterministic demand. Anily (1991) investigates the worst-case behavior of a heuristic for the multi-item replenishment and storage problem and derives a lower bound on the optimal average cost over all policies that follow stationary demand and cost parameters. Gallego et al. (1996) consider two economic order quantity models where multiple items use a common resource: the tactical and strategic models. They derive a lower bound on the peak resource usage that is valid for any feasible policy, use this to derive lower bounds on the optimum average cost for both models, and show that simple heuristics for either model have bounded worst-case performance ratios. Other literature, e.g., Rosenblatt and Rothblum (1990), Goyal (1978), Hartley and Thomas (1982), Jones and Inman (1989) and Dobson (1987)

studied deterministic inventory models with warehousing constraint.

Although much literature is devoted to multi-item, periodic-review systems with a production capacity constraint, little has been done for stochastic inventory models with a warehouse-capacity constraint. In particular, to our knowledge, no one has considered the case in which products from different manufacturers have distinct replenishment schedules or unequal lengths of replenishment intervals (we call this the case of unequal replenishment intervals). Unfortunately, except for very small problems, it is difficult to obtain an optimal replenishment policy in this case because we are unsure whether the objective cost function is convex. In this study, we develop three heuristics to estimate replenishment quantities for the case of unequal replenishment intervals.

We first characterize a stochastic multi-item inventory model for the case of equal replenishment intervals with limited warehouse capacity by using dynamic programming. We use induction to derive the convexity of objective cost function and determine the optimal replenishment policy for both a finite and an infinite planning horizon. From this derivation, we realize that when the warehousing constraint is tight, the overall warehouse capacity is divided into several terms each of which is related to products from an individual manufacturer, respectively, and is a constant. We consider each term as the capacity of the corresponding product. This result leads us to propose heuristics to determine replenishment policies for the case of unequal replenishment intervals.

In the first heuristic, we determine the capacity of each item separately and

apply this over the entire planning horizon. Given the separate capacity, this multi-item, unequal replenishment interval inventory model is reduced to multiple single item, equal replenishment interval problems and we are able to derive the optimal replenishment quantity of each item. The second heuristic is similar to the first one in the sense that we use a separate capacity for each item. Instead of using a fixed separate capacity for each item over the planning horizon, this heuristic adjusts individual capacities in each period according to the availability of any unused capacities of other items. Thus, compared to the first heuristic, it provides more room to those items that are to be replenished and we expect that this heuristic has better performance than the first one since it has more opportunity to replenish products. In the last heuristic, we use the total available capacity of the system at the moment to replenish products. Thus, this heuristic always maintains the greatest capacity for replenishment among the three proposed heuristics. We then compare the performance of each heuristic using numerical examples.

In supply chain management, limited resources in individual supply chain members critically affect modeling of the supply chain. So far, we studied a stochastic inventory model with a limited resource of warehousing capacity. We next discuss another stochastic system with limited resources. Consider a stochastic production system in which a single machine serves a series of customer orders to produce a single item. Because of a single production line coupled with a single machine, the amount of production in a given time is limited. Customer orders arrive in this single-stage, make-to-order production system, through which individual orders need to be

processed with their own due-date. Most of the due-date-setting models assume that the due dates for individual orders are set entirely exogenously. However, in certain practical contexts, each order arrives with a due date, indicating some future time when the customer wishes to receive the goods ordered and, in most cases, due date setting is negotiable and is the responsibility of the marketing personnel of the firm. For the marketing group, who have knowledge of customer's wishes, it is important to know their manufacturing personnel's perspective on the order before negotiating the due date with their customer. That is, they would like to know the optimal customer order leadtime to quote for the order. By *customer order leadtimes*, we mean the time from a customer's order until the due date.

Uncertainties in manufacturing environments play important roles in determining planned customer order leadtimes. These uncertainties are mostly due to complicated production processes, random yields, and high quality requirements. Incorporation of these factors into the selection of planned leadtimes represents an important step toward increasing the robustness of manufacturing planning. The leadtime uncertainty itself affects many aspects of costs and control. It is particularly problematic in queueing systems because tardiness in earlier orders may delay subsequent orders.

The flood of literature in this area focuses on leadtime uncertainties. Whybark and William (1976) compared safety stock and safety time as alternatives for buffering and found that safety time coped better with leadtime uncertainty. Melnyk and Piper (1985) examined the impact of PLT magnitude on manufacturing performance and found that larger PLTs improved on-time deliveries. St. John (1985) found that

larger PLTs increased total costs and supported the view that PLTs must be as small as possible. In particular, St. John showed that the carrying costapplied to raw materials, work in process, and finished goods inventorycontributed the greatest amount to the increase of the total costs. In contrast, Marlin (1986) suggested that inventory levels increased with the leadtime error magnitude rather than with the PLT magnitude. Marlin explained that this contradiction is due to shortcomings in the simulator used in St. John's study. Other literature such as Weeks (1981), Kanet (1986), Yano (1987a, b, c) and Moran (1998) studied empirical investigations of PLTs. Weeks proposed a one-period inventory model to set the optimal PLT and suggested that the PLT is a function of the cost of tardiness, earliness, due date length, and the manufacturing lead-time distribution. Weeks also showed that the single stage model with tardiness costs is equivalent to a simple newsboy problem.

Some efforts have focused on the impact of the leadtimes and their variability on optimal inventory decisions and system performance. The reduction of lead-times and their variability is a key element of process improvement, which in turn is at the core of the Just-In-Time approach (Schonberger 1982, 1986 and Shingo 1989). Porteus (1985, 1986) and Zangwill (1987) examined the effects of setup-cost reductions in deterministic, EOQ type production systems. Song and Zipkin (1996) investigated the joint effect of leadtime variance and lot sizes in a parallel processing environment. Hariharan and Zipkin (1995) extended basic inventory models to allow random demand leadtimes as well as random supply leadtimes. On the other hand, Duenyas (1995) formulated the problem of quoting optimal leadtimes as a semi-Markov decision process. He considered decision-making on leadtimes from the

customers' perspective instead of the common approach of considering the manufacturers' preference (Duenyas and Hopp 1995).

This study is stimulated by Yano's work on PLTs in serial production systems (1987a). Yano developed a solution approach to solve the two-stage problem and applied it to N-stage serial systems. The limitation of their approach was that they did not address the effects of queueing or capacity explicitly in the model. For instance, Yano assumes that an order that has completed processing in one stage may be held until its due date (i.e. no early delivery to customers is allowed). However, it is also assumed that once that order arrives at the next stage, it is processed immediately without being delayed by preceded orders. This assumption greatly simplifies their solution approach for determining PLTs in serial production systems. The assumption used by Yano is not pragmatic in many situations. Literature such as Duenyas (1995) and Duenyas and Hopp (1995) considered queueing effects in determining PLTs. In this study, we consider the aforementioned delay as well as the possible holding cost incurred due to the prohibition of early delivery to customers in single-stage systems.

This study addresses single-stage make-to-order (MTO) production systems. Our main objective is to determine the optimal planned customer leadtimes while minimizing the sum of expected inventory-holding costs and expected penalty costs for exceeding the quoted due date, so as to quote planned leadtimes to customers at the time of order arrival. Since the area of modeling leadtime setting where demand is sensitive to quoted leadtimes is relatively unexposed, it would not be a critical shortcoming to assume that the leadtimes quoted to customers are independent of

each other. Within this scope, we focus on modeling the appropriate expected costs under a variety modeling assumptions and characterizing the optimal policies.

Our approach to solving the problem is to derive the distribution of actual completion time of the process for an individual order and to compare this to the corresponding quoted due dates to obtain the expected total costs. We then minimize costs with respect to the decision variable(s), which, in this case, are PLT(s). We show that, in special cases, the single stage model is equivalent to a simple newsboy problem as Weeks (1981) pointed out.

Next is an outline of this dissertation. In Chapter 2, we introduce a stochastic multi-item inventory model for equal replenishment intervals with a limited warehouse capacity. We use induction to show the convexity of objective cost function and then determine the optimal replenishment policy in both a finite and an infinite planning horizon. In Chapter 3, we relax the assumption of identical replenishment schedules for items and characterize the corresponding inventory model. We derive the convexity of objective cost function and determine the optimal replenishment policy for the two-item, two-period problem as a special case. For large sized problems, we propose three heuristics to approximate the optimal replenishment quantities and compare the performance of each heuristic. Chapter 4 addresses the problem determining optimal planned leadtimes in single stage make-to-order production systems. We introduce an unique solution approach by deriving the distribution of the actual process completion time of newly arrived orders. In Chapter 5, we discuss conclusions from our study and briefly summarize future research directions.

CHAPTER 2
EQUAL REPLENISHMENT INTERVALS

2.1 Introduction

In this chapter, we study a stochastic multi-item inventory model with a warehouse-capacity constraint. The demand in each period of equal length is a random variable and inventories are reviewed periodically. Veinott (1965) first considered this dynamic nonstationary multi-item inventory model with discounted future costs. Veinott provided conditions that ensure that the base stock ordering policy is optimal and that the base stock levels in each period are easy to calculate. Ignall and Veinott (1969) show in stationary case that a myopic ordering policy is optimal for a sequence of periods for all initial inventory levels. Beyer et al. (2000a,b) take up a concrete version of the abstract model considered by Veinott (1965) and Ignall and Veinott (1969). Beyer et al. use the dynamic programming approach to derive the optimality of the ordering policy for the average cost problem and prove the convexity of the objective cost function. They present a detailed study of finite and infinite horizon multi-product, discounted- and average-cost problems. A part of their work is coincidentally indistinguishable from what we study in this chapter.

We first characterize a stochastic multi-item inventory model for the case of equal replenishment intervals with limited warehouse capacity by using dynamic programming. We then use induction to derive the convexity of objective cost function and determine the optimal replenishment policy for both a finite and an infinite planning

horizon.

2.2 Model Assumptions and Formulation

We first consider single-item inventory systems, facing limited warehouse capacity. A warehouse supplies single goods from a manufacturer to a population of customers. Customer demand in each period is an independently distributed random variable. When the demand cannot be met by inventory on hand, the order is backlogged until sufficient inventory is available. Distribution of the demand is assumed to be stationary. We assume that the leadtime between the manufacturer and the warehouse is deterministic and is set here as zero so that replenishment orders from the warehouse and corresponding shipments occur simultaneously. Further, a fixed schedule of shipments with equal intervals is preset at the beginning of each period. Therefore, the inventory system considered here is identical to periodic-review systems. As mentioned before, we do not assume any specific inventory policy such as the base stock policy. We use the following notation:

Parameters

m : number of replenishment periods,
j : index of replenishment periods, $j = 1, ..., m$,
V_0 : fixed storage capacity of the warehouse,
T : length of the time horizon,
h : unit cost of holding inventory at the warehouse,
p : unit penalty cost for demand backordered at the warehouse.

Demands

D_j : demand in period j, a random variable,

State and decision variables

I_j : inventory level at the beginning of period j before replenishment, where I_1 is given,

Q_j : size of replenishment in the jth period,

We define $g_j(I_j, Q_j)$ as the one-period expected holding and penalty costs in period j, given the initial inventory of I_j and the size of replenishment of Q_j. Then we have

$$\begin{aligned} g_j(I_j, Q_j) &= E[h(I_j + Q_j - D_j)^+ + p(D_j - I_j - Q_j)^+] \\ &= \int_{x=0}^{\infty} \left[h(I_j + Q_j - x)^+ + p(x - I_j - Q_j)^+\right] f_j(x)dx \\ &= h \int_{x=0}^{I_j+Q_j} (I_j + Q_j - x) f_j(x)dx + p \int_{x=I_j+Q_j}^{\infty} (x - I_j - Q_j) f_j(x)dx \end{aligned} \tag{2.1}$$

where f_j is the probability function of D_j.

To formulate this multi-period inventory model by using stochastic dynamic programming (SDP), we define I_j as the state variable of the system in period j. The decision in each period is to choose the size of replenishment, Q_j, for periods j through m, given a starting inventory of I_j. Given the initial state, the actions and the demands in period j, the state in the next period is determined recursively as follows:

$$I_{j+1} = I_j + Q_j - D_j, \quad j = 1, ..., m \tag{2.2}$$

With the definitions above we can state the recursive equations whose solution provides the optimal policy for the problem. Let $G_j(I_j, Q_j)$ be the expected holding

and penalty cost for periods j through m along with initial inventory of I_j. Also, define $G_j^*(I_j) = \min_{Q_j: I_j+Q_j \leq V_0} G_j(I_j, Q_j)$ as the minimum expected costs in period j through m, given that the system starts period j in state I_j and is subject to the capacity constraint $I_j + Q_j \leq V_0$. Thus, the following expression gives the expected cost, $G_j(I_j, Q_j)$, in period j through m:

$$\begin{aligned} G_j(I_j, Q_j) &= g_j(I_j, Q_j) + E\left[G_{j+1}^*(I_{j+1})\right] \\ &= g_j(I_j, Q_j) + \int_0^\infty \left[G_{j+1}^*(I_j + Q_j - D_j)\right] f(D_j) dD_j \\ &= g_j(I_j, Q_j) + \int_0^\infty \left[G_{j+1}(I_j + Q_j^* - D_j)\right] f(D_j) dD_j \qquad (2.3) \end{aligned}$$

where Q_j^* is the optimal solution in period j. The optimal cost for m periods then becomes $G_1^*(I_1)$ and can be calculated recursively in backward by, for $j = 2, ..., m$,

$$G_j^*(I_j) = \min_{Q_j: I_j+Q_j \leq V_0} \left[g_j(I_j, Q_j) + \int_0^\infty \left[G_{j+1}^*(I_j + Q_j - D_j)\right] f(D_j) dD_j\right]$$

Note that the capacity constraint is a linear function of Q_j. Thus, it is enough to show convexity of $G_j(I_j, Q_j)$ in Q_j in order to obtain the optimal solution of this constrained optimization problem. As we mentioned previously, we first show convexity of $G_m(I_m, Q_m)$ in the final period, determine the corresponding optimal solution, and then plug it in the expected cost function in the previous period $(m-1)$. We repeat this process until the beginning of the time horizon. Before we derive convexity of the m-period cost function, we first solve a single-period problem.

For simplicity, we drop the subscript j in a single-period and rewrite the problem as follows:

$$\min_{Q} \quad G(I,Q) = g(I,Q)$$
$$\text{subject to} \quad I + Q \leq V_0$$

Let $L = g(I,Q) + \mu(I + Q - V_0)$ where $\mu \geq 0$. Applying the well known Karush-Kuhn-Tucker (KKT) conditions, we obtain the following system of nonlinear algebraic equations and inequalities:

(a) $\frac{\partial L}{\partial Q} = 0$,

(b) $I + Q \leq V_0$,

(c) $\mu(I + Q - V_0) = 0$,

(d) $\mu \geq 0$,

Before we solve these equations, we introduce the rule for differentiating integrals called *Leibnitz's Rule* (Casella and Berger (1990)).

LEIBNITZ'S RULE : If $f(x,\theta), a(\theta)$ and $b(\theta)$ are differentiable with respect to θ, then

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta}f(x,\theta)dx.$$

Notice that if $a(\theta)$ and $b(\theta)$ are constants, we have a special case of *Leibnitz's Rule*:

$$\frac{d}{d\theta}\int_{a}^{b} f(x,\theta)dx = \int_{a}^{b} \frac{\partial}{\partial\theta}f(x,\theta)dx.$$

Any interchangeability of integral and partial derivative in this study follows this rule. From (2.1) and (a), we have

$$\begin{aligned}
\frac{\partial L}{\partial Q} &= \frac{\partial}{\partial Q} g(I,Q) + \frac{\partial}{\partial Q}\left[\mu(I+Q-V_0)\right] \\
&= \frac{\partial}{\partial Q}\left[h\int_0^{I+Q}(I+Q-D)f(D)dD + p\int_{I+Q}^{\infty}(D-I-Q)f(D)dD\right] \\
&\qquad + \frac{\partial}{\partial Q}\left[\mu(I+Q-V_0)\right] \\
&= h\int_0^{I+Q} f(D)dD - p\int_{I+Q}^{\infty} f(D)dD + \mu \\
&= hF(I+Q) - p(1-F(I+Q)) + \mu \\
&= (h+p)F(I+Q) - p + \mu = 0 \qquad (2.4)
\end{aligned}$$

where $F(\cdot)$ is the cumulative distribution function of D. By taking the second derivative on L, we get, from (2.4),

$$\begin{aligned}
\frac{\partial^2 L}{\partial Q^2} &= \frac{\partial^2}{\partial Q^2} g(I,Q) + \frac{\partial^2}{\partial Q^2}\left[\mu(I+Q-V_0)\right] \\
&= (h+p)f(I+Q) \qquad (2.5)
\end{aligned}$$

Since the right hand side in (2.5) is always nonnegative, the expected cost for a single-period is convex in the replenishment size, Q. The following two propositions are immediate from above.

Proposition 2.1 *The expected holding and penalty cost for a single-period, $G(I,Q)$, with a limited capacity is a convex function of the replenishment size, Q.*

Proposition 2.2 $Q^* = min\left[F^{-1}\left(\frac{p}{h+p}\right), V_0\right] - I$ *is the optimal replenishment policy in a single-period problem*

Proof : Solving (2.4) and (b)-(d), we get Q^* (See Appendix A for details). □

Now, we show convexity of $G_j(I_j, Q_j)$ in Q_j. Assuming $G_j(I_j, Q_j)$ is twice differentiable, we can then derive the following:

Proposition 2.3 *The expected holding and penalty cost for periods j through m, $G_j(I_j, Q_j)$, is convex in the replenishment size in period j, Q_j.*

Proof : First, we take the first and second partial derivatives of $G_j(I_j, Q_j)$ with respect to Q_j from (2.3):

$$\frac{\partial G_j(I_j, Q_j)}{\partial Q_j} = \frac{\partial g_j(I_j, Q_j)}{\partial Q_j} + \int_0^\infty \frac{\partial}{\partial Q_j}\left[G^*_{j+1}(I_{j+1})\right] f_j(x)dx \tag{2.6}$$

$$\frac{\partial^2 G_j(I_j, Q_j)}{\partial {Q_j}^2} = \frac{\partial^2 g_j(I_j, Q_j)}{\partial {Q_j}^2} + \int_0^\infty \frac{\partial^2}{\partial {Q_j}^2}\left[G^*_{j+1}(I_{j+1})\right] f_j(x)dx \tag{2.7}$$

The first term in the right-hand side in (2.7) is equivalent to the second derivative of expected cost for a single-period problem, and thus it is always nonnegative, referring to Proposition 2.1.

Now, we show, using induction, that the second term in (2.7) is nonnegative. First, consider period m. Note that this SDP problem in period m is identical to the single period problem described previously because $G^*_{m+1}(I_{m+1}) = 0$. That is, the decision in the final period m is identical to that for a single period problem since the recursive part becomes zero. Since the first and second derivatives of $G^*_{m+1}(I_{m+1})$ are also zero, it is obvious, from Proposition 2.1, that $G_m(I_m, Q_m)$ is convex in Q_m. In addition, we can determine the optimal solution in period m, based on the result from the Proposition 2.2, as follows:

$$Q_m^* = min\left[F_m^{-1}\left(\frac{p}{h+p}\right), V_0\right] - I_m \tag{2.8}$$

Next, we assume that, for period $(k+1)$, $G_{k+1}(I_{k+1}, Q_{k+1})$ is convex in Q_{k+1} and the optimal solution in period $(k+1)$ is

$$Q_{k+1}^* = min\left[F_{k+1}^{-1}\left(\frac{p}{h+p}\right), V_0\right] - I_{k+1}$$

Substituting Q_{k+1}^* into the expected cost in period $(k+1)$, we get

$$\begin{aligned}
E\left[G_{k+1}^*(I_{k+1})\right] &= E\left[h\left(I_{k+1} + Q_{k+1}^* - D_{k+1}\right)^+ + p\left(D_{k+1} - I_{k+1} - Q_{k+1}^*\right)^+\right] \\
&= E\left[h\left(min\left[F_{k+1}^{-1}\left(\frac{p}{h+p}\right), V_0\right] - D_{k+1}\right)^+\right. \\
&\qquad \left. + p\left(D_{k+1} - min\left[F_{k+1}^{-1}\left(\frac{p}{h+p}\right), V_0\right]\right)^+\right]
\end{aligned}$$

We can notice that the optimal expected cost in period $(k+1)$ is not a function of decision variables any more. That is, $E\left[G_{k+1}^*(I_{k+1})\right]$ is a constant over any $Q_j, j = 1, ..., m$.

We now show convexity of $G_k(I_k, Q_k)$ in Q_k. Taking the first and second partial derivatives on the expected cost in period k,

$$\begin{aligned}
\frac{\partial G_k(I_k, Q_k)}{\partial Q_k} &= \frac{\partial}{\partial Q_k} g_k(I_k, Q_k) + \frac{\partial}{\partial Q_k} E\left[G_{k+1}^*(I_{k+1})\right] \\
&= \frac{\partial}{\partial Q_k} g_k(I_k, Q_k) \qquad \left(\because \frac{\partial}{\partial Q_k} E\left[G_{k+1}^*(I_{k+1})\right] = 0\right) \\
&= (h+p)F_k(I_k + Q_k) - p \qquad \text{(from (2.1) and (2.4))} \\
\frac{\partial^2 G_k(I_k, Q_k)}{\partial {Q_k}^2} &= (h+p) f_k(I_k + Q_k)
\end{aligned}$$

and the right hand side in the second derivative is always nonnegative. Therefore, $G_k(I_k, Q_k)$ is a convex function of Q_k. Finally, by induction, we conclude that, for any period j, $G_j(I_j, Q_j)$ is convex in Q_j. $\square$

The result in Proposition 2.3 shows that the solution satisfying the first-order condition, (2.6), minimizes the expected holding and penalty costs for periods j through m, $G_j(I_j, Q_j)$, and we can easily obtain the optimal solution in period j, analogous to (2.8):

Proposition 2.4 $Q_j^* = min\left[F_j^{-1}\left(\frac{p}{h+p}\right), V_0\right] - I_j$ *is the optimal replenishment policy in period j, where I_j is the initial inventory at the beginning of period j,* $j = 1, ..., m$.

Alternatively, we can express the optimal solution above as follows: $Q_j^* = F_j^{-1}[(p - \mu^*)/(h + p)] - I_j$, where $\mu^* = arg \min_{\mu \geq 0} F_j^{-1}[(p - \mu)/(h + p)] \leq V_0$. According to Proposition 2.4, the optimal replenishment quantity in one period is a function of its initial inventory as well as the distribution function of its demand and the storage capacity. Furthermore, we can see that, referring to (2.3) and Proposition 2.4, the recursive term in the expected cost is a constant over the current decision variable, Q_j. That is, the future cost incurred in that period is not affected by the replenishment quantity determined at the beginning of the period. This result gives us a good intuition when we extend this finite horizon, periodic-review model to the infinite horizon case because deriving convexity of the expected cost function

in any period is not affected by how many replenishment periods are remained in the future. We will discuss this issue in detail later.

2.3 Multi-Product Inventory Systems

In this section, we extend the single-item inventory system to the multi-item case. Consider a multi-item inventory system, in which a warehouse serves manufacturers by storing their products and delivering them to their respective customers. Each manufacturer produces a single item which is shipped to the warehouse according to its preset schedule. The replenishment leadtimes between the warehouse and manufacturers are assumed to be zero so that the replenishment orders from the warehouse to manufacturers and the corresponding shipments occur simultaneously, and all replenishment are received at the beginning of the following replenishment period. This situation might occur in a JIT environment when suppliers are located in close proximity to a warehouse.

Again, we assume that items are replenished periodically by manufacturers. Customer demand for items in each period is an independently distributed random variable, and is backlogged when it cannot be met by inventory on hand until sufficient inventory is available. The distribution of the demand is assumed to be stationary. Most of the notation used here is analogous to that in the single item case except for the extra index for items as follows:

Parameters

l : number of product,

h_i : unit cost of holding inventory for item i,
p_i : unit penalty cost of item i for demand backordered at the warehouse.

Demands

$D_{i,j}$: demand of item i in period j, a random variable,

State and decision variables

$I_{i,j}$: inventory level of item i at the beginning of period j before replenishment, where $I_{i,1}$ is given,
$Q_{i,j}$: size of the replenishment of item i in the jth period, $i = 1, ..., l$,

We first define $S_j = (I_{1,j}, ..., I_{l,j})$ as a vector of state variables of the system in period j and let $Q_j = (Q_{1,j}, ..., Q_{l,j})$ be a vector of decision variables. For a given state S_j, the current inventory level of the system in period j is $I_j = \sum_{i=1}^{l} I_{i,j}$, which can be expressed recursively as follows:

$$I_j = I_{j-1} + \sum_{i=1}^{l} Q_{i,j-1} - \sum_{i=1}^{l} D_{i,j}, \ j = 2, .., m \tag{2.9}$$

We also can determine the current inventory levels of individual items recursively using the state in the previous period:

$$I_{i,j} = I_{i,j-1} + Q_{i,j-1} - D_{i,j}, \ j = 2, ..., m \tag{2.10}$$

We define $g_j(S_j, Q_j)$ as the expected holding and penalty costs of the system in period j, given initial inventories of $S_j = (I_{1,j}, ..., I_{l,j})$ and replenishment sizes of $Q_j = (Q_{1,j}, ..., Q_{l,j})$. Then we have

$$g_j(S_j, Q_j) \ = \ \sum_{i=1}^{l} E\left[h_i(I_{i,j} + Q_{i,j} - D_{i,j})^+ + p_i(D_{i,j} - I_{i,j} - Q_{i,j})^+\right] \tag{2.11}$$

We state the recursive equations whose solution provides the optimal policy for the problem. Denote the expected holding and penalty costs for periods j through m by $G_j(S_j, Q_j)$. Then $G_j(S_j, Q_j)$ can be expressed as follows:

$$\begin{aligned} G_j(S_j, Q_j) &= g_j(S_j, Q_j) + E\left[G^*_{j+1}(S_{j+1})\right] \\ &= g_j(S_j, Q_j) + \int_0^\infty \cdots \int_0^\infty \left[G^*_{j+1}(S_{j+1})\right] f_{1,j}(x_1) \cdots f_{l,j}(x_l) dx_1 \cdots dx_l \end{aligned} \tag{2.12}$$

where $f_{i,j}$ is the probability density function of a random demand, $D_{i,j}$, $G^*_k(S_k) = \min_{Q_{i,k}:\sum_{i=1}^{l}(I_{i,k}+Q_{i,k})\leq V_0} G_k(S_k, Q_k)$ is the minimum expected costs in period k through m, $_{k=1,\dots,m}$, given the system starts period k in state $S_k = (I_{1,k}, ..., I_{l,k})$ and is subject to the capacity constraint. We eventually want to find the optimal cost for a whole planning horizon, $G^*_1(I_1)$, which can be calculated recursively in backward by, for $j = 2, ..., m$,

$$\begin{aligned} G^*_j(S_j) = \min_{Q_{i,j}:\sum_{i=1}^{l}(I_{i,j}+Q_{i,j})\leq V_0} & \left[g_j(S_j, Q_j) + \int_0^\infty \cdots \right. \\ & \left. \int_0^\infty \left[G^*_{j+1}(S_{j+1})\right] f_{1,j}(x_1) \cdots f_{l,j}(x_l) dx_1 \cdots dx_l \right] \end{aligned}$$

As in the single-item case, the capacity constraint is a linear function of $Q_{i,j}$ and thus it is enough to show convexity of $G_j(S_j, Q_j)$ to get the optimal solution of above SDP problem. Applying the same induction method as in the single-item case, we have the following two propositions.

Proposition 2.5 *The expected holding and penalty cost for periods j through m, $G_j(S_j, Q_j)$, is convex in replenishment sizes in period j, $Q_{i,j}$ where $Q_j = (Q_{1,j}, ..., Q_{l,j})$.*

Proof : Taking the first and second partial derivatives of (2.12) yields followings: For any $i, i' = 1, ..., n$,

$$\begin{aligned}\frac{\partial G_j(S_j, Q_{1,j}, ..., Q_{l,j})}{\partial Q_{i,j}} &= \frac{\partial}{\partial Q_{i,j}} g_j(S_j, Q_{1,j}, ..., Q_{l,j}) \\ &+ \int_0^\infty \cdots \int_0^\infty \frac{\partial G^*_{j+1}}{\partial Q_{i,j}} f_{1,j}(x_1) \cdots f_{l,j}(x_l) dx_1 \cdots dx_l\end{aligned} \tag{2.13}$$

$$\begin{aligned}\frac{\partial^2 G_j(S_j, Q_{1,j}, ..., Q_{l,j})}{\partial {Q_{i,j}}^2} &= \frac{\partial^2}{\partial {Q_{i,j}}^2} g_j(S_j, Q_{1,j}, ..., Q_{l,j}) \\ &+ \int_0^\infty \cdots \int_0^\infty \frac{\partial^2 G^*_{j+1}}{\partial {Q_{i,j}}^2} f_{1,j}(x_1) \cdots f_{l,j}(x_l) dx_1 \cdots dx_l\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 G_j(S_j, Q_{1,j}, ..., Q_{l,j})}{\partial Q_{i,j} \partial Q_{i',j}} &= \frac{\partial^2}{\partial Q_{i,j} \partial Q_{i',j}} g_j(S_j, Q_{1,j}, ..., Q_{l,j}) \\ &+ \int_0^\infty \cdots \int_0^\infty \frac{\partial^2 G^*_{j+1}}{\partial Q_{i,j} \partial Q_{i',j}} f_{1,j}(x_1) \cdots f_{l,j}(x_l) dx_1 \cdots dx_l\end{aligned}$$

The interchangeability of integral and partial derivative above is based on *Leibnitz'* Rule. We use Hessian matrices to express above second derivatives. Define:

$$H_j = \begin{bmatrix} \partial^2 G_j / \partial {Q_{1,j}}^2 & \cdots & \partial^2 G_j / \partial Q_{1,j} \partial Q_{l,j} \\ \cdots & \partial^2 G_j / \partial Q_{i,j} \partial Q_{i',j} & \cdots \\ \partial^2 G_j / \partial Q_{l,j} \partial Q_{1,j} & \cdots & \partial^2 G_j / \partial {Q_{l,j}}^2 \end{bmatrix}$$

$$B_j = \begin{bmatrix} \partial^2 g_j/\partial {Q_{1,j}}^2 & \cdots & \partial^2 g_j/\partial Q_{1,j}\partial Q_{l,j} \\ \cdots & \partial^2 g_j/\partial Q_{i,j}\partial Q_{i',j} & \cdots \\ \partial^2 g_j/\partial Q_{l,j}\partial Q_{1,j} & \cdots & \partial^2 g_j/\partial {Q_{l,j}}^2 \end{bmatrix}$$

$$E_{j+1} = \begin{bmatrix} \partial^2 G^*_{j+1}/\partial {Q_{1,j}}^2 & \cdots & \partial^2 G^*_{j+1}/\partial I_{1,j+1}\partial Q_{l,j} \\ \cdots & \partial^2 G^*_{j+1}/\partial Q_{i,j}\partial Q_{i',j} & \cdots \\ \partial^2 G^*_{j+1}/\partial Q_{l,j}\partial Q_{1,j} & \cdots & \partial^2 G^*_{j+1}/\partial {Q_{l,j}}^2 \end{bmatrix}$$

Then Hessian matrices defined above have the following relationship, based on (2.12):

$$H_j = B_j + \int_0^\infty \cdots \int_0^\infty [E_{j+1}] f_{1,j}(x_1) \cdots f_{l,j}(x_l) dx_1 \cdots dx_l \tag{2.14}$$

Note that B_j is equivalent to Hessian of a single-period expected cost, and is always positive semi-definite in any period j (see Appendix B for details). We use induction to show that H_j is also positive semi-definite. Consider the final period m. Assuming $G^*_{m+1}(S_{m+1}) = 0$, it is obvious, by definition, that E_{m+1} is a zero matrix and $H_m = B_m$. Therefore, H_m is positive semi-definite because B_m is, or $G_m(S_m, Q_m)$ is a convex function in $Q_{i,m}$. Furthermore, we can determine the optimal solutions in period m using the convexity of G_m (see Appendix C for details):

$$Q^*_{i,m} = F^{-1}_{i,m}\left(\frac{p_i - \mu^*_m}{h_i + p_i}\right) - I_{i,m},$$

where μ_m is a Lagrange Multiplier, $F_{i,m}$ is the cumulative distribution function of f_i, and

$$\mu^*_m = arg \min_{\mu_m \geq 0} \left[\sum_{i=1}^{l} F^{-1}_{i,m}\left(\frac{p_i - \mu_m}{h_i + p_i}\right) \leq V_0\right]$$

We assume that H_{k+1} is positive semi-definite, and that the optimal solution in period $(k+1)$ is

$$Q^*_{i,k+1} = F^{-1}_{i,k+1}\left(\frac{p_i - \mu^*_{k+1}}{h_i + p_i}\right) - I_{i,k+1},$$

where

$$\mu^*_{k+1} = arg \min_{\mu_{k+1}\geq 0}\left[\sum_{i=1}^{l} F^{-1}_{i,k+1}\left(\frac{p_i - \mu_{k+1}}{h_i + p_i}\right) \leq V_0\right]$$

Then the optimal expected cost in period $(k+1)$ can be obtained by substituting Q_{k+1} by Q^*_{k+1} in $G_{k+1}(S_{k+1}, Q_{k+1})$.

$$\begin{aligned}
G^*_{k+1}(S_{k+1}) &= G_{k+1}(S_{k+1}, Q^*_{k+1}) \\
&= \sum_{i=1}^{l} E\Big[h_i\left(I_{i,k+1} + Q^*_{i,k+1} - D_{i,k+1}\right)^+ \\
&\qquad + p_i\left(D_{i,k+1} - I_{i,k+1} - Q^*_{i,k+1}\right)^+\Big] \qquad (2.15) \\
&= \sum_{i=1}^{l} E\left[h_i\left(F^{-1}_{i,k+1}\left(\frac{p_i - \mu^*_{k+1}}{h_i + p_i}\right) - D_{i,k+1}\right)^+\right. \\
&\qquad \left. + p_i\left(D_{i,k+1} - F^{-1}_{i,k+1}\left(\frac{p_i - \mu^*_{k+1}}{h_i + p_i}\right)\right)^+\right] \qquad (2.16)
\end{aligned}$$

We can see that the optimal expected cost, $G^*_{k+1}(I_{k+1})$ is a constant over any decision variables. We now show positive semi-definiteness of H_k. From (2.12), we get the following:

$$\begin{aligned}
\frac{\partial}{\partial Q_{i,k}} G_k(S_k, Q_{1,k}, ..., Q_{l,k}) &= \frac{\partial}{\partial Q_{i,k}} g_k(S_k, Q_{1,k}, ..., Q_{l,k}) \\
&\qquad + \frac{\partial}{\partial Q_{i,k}} E\left[G^*_{k+1}(S_{k+1})\right] \\
&= \frac{\partial}{\partial Q_{i,k}} g_k(S_k, Q_{1,k}, ..., Q_{l,k}) \\
&\qquad (\because E\left[G^*_{k+1}(S_{k+1})\right] = 0)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial Q_{i,k}^2} G_k(S_k, Q_{1,k}, ..., Q_{l,k}) &= \frac{\partial^2}{\partial Q_{i,k}{}^2} g_k(S_k, Q_{1,k}, ..., Q_{l,k}) \\
\frac{\partial^2}{\partial Q_{i,k} \partial Q_{i',k}} G_k(S_k, Q_{1,k}, ..., Q_{l,k}) &= \frac{\partial^2}{\partial Q_{i,k} \partial Q_{i',k}} g_k(S_k, Q_{1,k}, ..., Q_{l,k})
\end{aligned}$$

By definition, $H_k = B_k$ That is, H_k is positive semi-definiteness because B_k is. Therefore, we can conclude that H_j is convex in $Q_{i,j}$ for any period j, $j = 1, ..., m$. □

The result in Proposition 2.5 shows that the solution satisfying the first order condition is a unique optimal solution for item i in period j. The solution approach in here is analogous to that in the single-item case since the expected cost is a linear function of decision variables (see (2.11)), and the recursive term is a constant over those variables. The following proposition states the optimal replenishment policy for item i in period j in multi-item, periodic-review inventory system:

Proposition 2.6 *The optimal replenishment policy for item i in period j is*

$$Q_{i,j}^* = F_{i,j}^{-1}\left(\frac{p_i - \mu_j^*}{h_i + p_i}\right) - I_{i,j} \tag{2.17}$$

where μ_j is a Lagrange Multiplier, $F_{i,j}$ is the cumulative distribution function of a random demand $D_{i,j}$, and

$$\mu_j^* = arg \min_{\mu_j \geq 0} \left[\sum_{i=1}^{l} F_{i,j}^{-1}\left(\frac{p_i - \mu_j}{h_i + p_i}\right) \leq V_0 \right] \tag{2.18}$$

According to Proposition 2.6, the optimal replenishment quantity of an individual item in one period is a function of its initial inventory as well as the distribution function of its demand and the storage capacity. Furthermore, Proposition 2.6 leads us to state that, in any period j, $Q^*_{i,j} + I_{i,j}$, which is the optimal inventory level of item i after replenishment, is a constant over the decision variable in that period, $Q_{i,j}$. Therefore, as we have seen in (2.15) and (2.16), after replacing $Q^*_{i,j} + I_{i,j}$ by $F_j^{-1}(\cdot)$, the recursive term in (2.12), $G^*_{j+1}(S_{j+1})$, becomes independent of $Q_{i,j}$. When we derive the convexity of the expected cost function in any period j, $G_j(S_j, Q_j)$, we then only need to show the convexity of the single-period expected cost function, $g_j(S_j, Q_j)$ because the recursive term, $E[G^*_{j+1}(S_{j+1})]$, is a constant over $Q_{i,j}$. And deriving the convexity of $g_j(S_j, Q_j)$ is usually not difficult, according to Appendix C. As we mentioned in the single-item case, this result would be valuable when we extend this finite horizon, periodic-review model to a infinite horizon problem. Before we move to an infinite horizon case, we state the following theorem based on above arguments.

Theorem 2.1 *The optimal replenishment quantity in a single period is independent of the future expected cost incurred in that period*

2.4 Numerical Results

In this section, we illustrate numerical solutions of the optimal replenishment policy obtained in (2.17) for various cost parameters and capacities.

Exponentially distributed demand

If demand of item i in period j is exponentially distributed with a mean $1/\lambda_{i,j}$, the cumulative distribution function of the demand is as follows:

$$\begin{aligned} F_{i,j}(x) &= \int_0^x \frac{1}{\lambda_{i,j}} e^{-\frac{s}{\lambda_{i,j}}} ds \\ &= 1 - e^{-\frac{x}{\lambda_{i,j}}} \end{aligned}$$

and the inverse of $F_{i,j}(x)$ is

$$F_{i,j}^{-1}(y) = -\lambda_{i,j} \ \ log(1-y)$$

From (2.14) and (2.15), we get

$$\begin{aligned} Q_{i,j}^* &= -\lambda_{i,j} \ \ log\left(\frac{h_i + \mu_j^*}{h_i + p_i}\right) - I_{i,j} \\ \mu_j^* &= arg \min_{\mu_j \geq 0} \left[\sum_{i=1}^{l} -\lambda_{i,j} \ \ log\left(\frac{h_i + \mu_j}{h_i + p_i}\right) \leq V_0\right] \end{aligned}$$

Numerical solutions for $Q_{i,j}$ in above equations can be obtained from the following example.

Example.

Let $l = 2$ and $j = 1$. Let the mean demand of each item for one period be $\lambda_{1,1} = 100$, $\lambda_{2,1} = 120$, respectively, and the capacity limit is $V_0 = 200$. Assume that the initial inventory levels are $I_{1,1} = 10$ and $I_{2,1} = 20$, and the holding and penalty costs are $h_1 = 5, h_2 = 5, p_1 = 10$, and $p_2 = 10$. Then we get $\mu_j^* = 1.04$. Thus, the optimal replenishment quantity of each item is $Q_{1,1}^* = 81$ and $Q_{2,1}^* = 89$.

1. *Effect of capacity on replenishment quantity.*

Figures 2.1 and 2.2 represent the relationship between replenishment quantities of individual items and warehouse capacities for different unit costs. Not surprisingly, these figures indicate that the replenishment quantity of each item increases up to a certain level as capacity increases and then becomes flat afterward for any combination of cost parameters. That is, at a certain level of capacity the capacity constraint is not binding anymore to the problem.

We also notice that, in Figure 2.1, item 1 has more replenishment quantity than item 2 when the capacity decreases, although the latter has a greater mean demand. This is because when the size of capacity is small, the difference in the optimal inventory level (after replenishment) of each item would not exceed the difference in the initial inventory level of each item, so that item 2 is replenished less.

2. *Lower bound of capacity in each period.*

The replenishment quantities of items are stabilized at a certain level of capacity (250 in Figure 2.1 and 210 in Figures 2.2, which are the lower bounds of the capacity for each case). By specifying input parameters and running a simulation, the lower bound of capacity in each period can be easily determined. It is advantageous to warehouse managements who want to utilize their storage more effectively over periods.

2.5 Infinite Horizon Case

We now extend to a infinite horizon case. While we label periods as time to the end of the planning horizon in the finite horizon problem, it is typical that the process

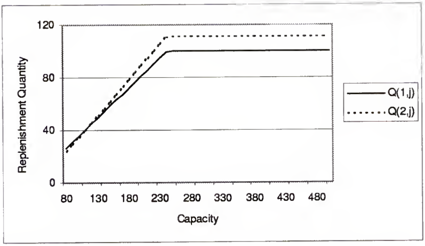


Figure 2.1: Optimal replenishment quantities versus capacity ($h_1 = 5, p_1 = 10, h_2 = 5, p_2 = 10$)

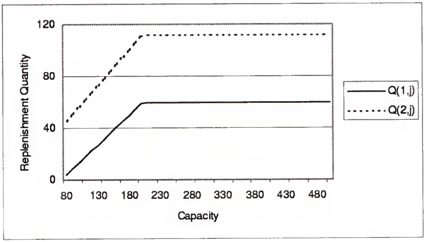


Figure 2.2: Optimal replenishment quantities versus capacity ($h_1 = 5, p_1 = 5, h_2 = 5, p_2 = 10$)

starts at some point and continues indefinitely in the infinite horizon case. Analytical issues such as existence and convergence of the infinite horizon cost, and optimality of stationary policies have explored in many papers such as Federgruen and Zipkin (1986a, b), DeCroix and Arreola-Risa (1998), and Kapuscinski and Tayur (1998). Especially, Federgruen and Zipkin (1987b) show that, under certain assumptions, every infinite horizon policy has finite expected discounted cost in single-product case. DeCroix and Arreola-Risa (1998) argue that every finite expected discounted cost in multi-item case is bounded above, and show the existence of the optimal cost in infinite horizon cases. We do not explore detailed analytical issues in this study. Instead, we show the convexity of the expected cost function for the infinite horizon case, and determine the optimal replenishment policy.

Let β is the discounted factor, where $0 < \beta < 1$. Then the finite horizon cost function in period n is

$$G_n(S_n, Q_n) = g_n(S_n, Q_n) + \beta E\left[G^*_{n+1}(S_{n+1})\right], \; n = 1, ..., m \tag{2.19}$$

where

$$G^*_n(S_n) = \min_{Q_{i,n}:\; \mathbf{1}^T(S_n+Q_n)\leq V_0} \left[g_n(S_n, Q_n) + \beta E\left[G^*_{n+1}(S_{n+1})\right]\right]$$

and $G^*_{m+1}(S_{m+1}) = 0$, $\mathbf{1}^T = (1, ..., 1)$.

Define $Z_n(S'_n, Q'_n) = G_{m-n+1}(S_{m-n+1}, Q_{m+n-1})$ and $z_n(S'_n, Q'_n) = g_{m-n+1}(S_{m-n+1}, Q_{m+n-1})$. Then $G_n(S_n, Q_n)$ can be expressed as follows: For $n = 1, ..., m$

$$Z_n(S'_n, Q'_n) = z_n(S'_n, Q'_n) + \beta E\left[Z^*_{n-1}(S'_n + Q'_n - D_n)\right] \tag{2.20}$$

where $D_n = (D_{1,n}, ..., D_{l,n})$ is a vector of random demands and $Z^*_0(S'_0) = 0$.

Let $m = \infty$ and $n \to \infty$. Then the second term in the right hand side in (2.20) is iterated recursively for indefinite number of periods in order to compute the infinite horizon expected cost.

We now show the convexity of $Z_n(S'_n, Q'_n)$ as $n \to \infty$. Following the arguments of DeCroix and Arreola-Risa (1998), every $Z^*_n(S'_n)$ is bounded above and there exists a function $Z^*(S')$ with

$$Z^*(S') = \lim_{n\to\infty} Z_n{}^*(S'_n)$$

since $Z^*_n(S'_n)$ is nonnegative and thus is increasing in n for fixed S'_n. That is, the infinite horizon cost is convergent and is well defined. Analogous to above notation, let $Z(S') = \lim_{n\to\infty} Z_n(S'_n, Q'_n)$ and let Q^∞_n be the optimal solution in infinite horizon case, starting in period n. We then have the following proposition:

Proposition 2.7 *The expected holding and penalty cost for periods n through ∞, $Z(S')$ is convex in replenishment sizes, Q'_n where $Q'_n = (Q_{1,n}, ..., Q_{l,n})$.*

Proof : It is straightforward to derive the convexity of $Z(S')$. From (2.20), the optimal finite horizon cost is

$$Z^*_n(S'_n) = z_n(S'_n, Q'_n) + \beta E\left[Z^*_{n-1}(S'_n + Q'_n - D_n)\right],\ n = 1, ..., m$$

and when $m = \infty$ and $n \to \infty$, we have

$$Z^*(S') = z_n(S'_n, Q^\infty_n) + \beta E\left[Z^*_{n-1}(S'_n + Q^\infty_n - D_n)\right]$$

As we argued on Theorem 2.1, the recursive term or the future cost, Z^*_{n-1}, is always independent of decision variables whether n is a finite or goes to infinity. Therefore, the corresponding term is vanished in the first and second partial derivatives with respect to $Q'_n = Q^\infty_n$, and our task reduces to derive the convexity of the single-period cost, $z_n(S'_n, Q^\infty_n)$, which is obvious from the previous result. Therefore $Z^*(S')$ is convex and $Z^*(S') \to \infty$ as $\|S'\| \to \infty$. As a result, $Z(S')$ is convex and $Z(S') \to \infty$ as $\|S'\| \to \infty$ $\square$

DeCroix and Arreola-Risa (1998) showed the convexity of the infinite horizon discounted cost in multi-item, periodic review inventory system under the assumption of homogeneous products. They also argued that, in case of non-homogeneous products, they could not provide a practical approach for solving the problem. The result of Proposition 2.7 is attractive because, in this study, we did not restrict our assumption to the homogeneous product case. We also can notice that our method to derive the convexity of the expected cost function in either a finite or an infinite horizon case is straightforward and easy to understand. The following proposition provide the optimal solution of the infinite horizon case.

Proposition 2.8 *The optimal replenishment policy for item i in the infinite horizon case is*

$$Q_n^{\infty} = F_{i,n}^{-1}\left(\frac{p_i - \mu^{\infty}}{h_i + p_i}\right) - I_{i,n}, \ n = 1, 2, ...$$

where μ is a Lagrange Multiplier, $F_{i,n}$ is the cumulative distribution function of a random demand $D_{i,n}$, and

$$\mu^{\infty} = arg \min_{\mu \geq 0} \left[\sum_{i=1}^{l} F_{i,n}^{-1}\left(\frac{p_i - \mu}{h_i + p_i}\right) \leq V_0\right]$$

Proof: Define $L = G_n(I_{1,n}, ..., I_{l,n}, Q_{1,n}, ..., Q_{l,n}) + \mu\left(\sum_{i=1}^{l} I_{i,n} + \sum_{i=1}^{l} Q_{i,n} - V_0\right)$. Then, from (2.20), we get

$$\begin{aligned}
\frac{\partial L}{\partial Q_{i,n}} &= \frac{\partial}{\partial Q_{i,n}} G_n + \frac{\partial}{\partial Q_{i,n}}\left[\mu(\sum_{i=1}^{l} I_{i,n} + \sum_{i=1}^{l} Q_{i,n} - V_0)\right] \\
&= \frac{\partial}{\partial Q_{i,n}} g_n + \mu \\
&= (h_i + p_i) F_{i,n}(I_{i,n} + Q_{i,n}) - p_i + \mu = 0 \qquad (2.21)
\end{aligned}$$

Solving (2.21) and the KKT condition: $\mu(\sum_{i=1}^{l} I_{i,n} + \sum_{i=1}^{l} Q_{i,n} - V_0) = 0$, we get

$$\begin{aligned}
Q_{i,n}^{\infty} &= F_{i,n}^{-1}\left(\frac{p_i + \mu^{\infty}}{h_i + p_i}\right) - I_{i,n} \\
\mu^{\infty} &= arg \min_{\mu \geq 0} \left[\sum_{i=1}^{l} F_{i,n}^{-1}\left(\frac{p_i - \mu}{h_i + p_i}\right) \leq V_0\right] \quad \square
\end{aligned}$$

2.6 Conclusions

In this chapter, we discuss a stochastic multi-item inventory model for the case of equal replenishment intervals with a warehouse-capacity constraint. We completely characterize the inventory model by using dynamic programming. We use

induction to derive the convexity of objective cost function and show that a myopic replenishment policy is optimal for both a finite and an infinite planning horizon. Furthermore, we notice that the optimal policy in each period is easy to calculate due to the independence of the one period expected cost over periods.

CHAPTER 3
UNEQUAL REPLENISHMENT INTERVALS

3.1 Introduction

In this chapter, we extend our study to a case where individual items have different replenishment schedules. By relaxing the assumption of identical replenishment schedules for items we generalize the multi-item, periodic-review inventory model described in the previous chapter. Items are replenished periodically by manufacturers, but individual items have different replenishment schedules. Thus, the simple periodic-review inventory policy is not valid anymore in this case. This assumption can be partly explained as follows: The transportation company that controls the entire distribution system can efficiently utilize its transportation resources and procure its transportation services at lower cost by assigning a different shipment schedule to each product. A warehouse supplies products from manufacturers to a population of customers. We first state the model assumptions and then formulate the expected cost function to determine an optimal replenishment policy

3.2 Model Assumptions and Formulation

Customer demands for items are random variables, and backlogging is allowed. The distribution of demand for each product is assumed to be stationary. We assume zero replenishment leadtime between the warehouse and the manufacturers, and all replenishment orders are received at the beginning of the following replenishment

period. A finite planning horizon is considered. We use the following notation to describe this capacity constrained multi-item inventory model with different replenishment schedules:

Parameters

l : number of items,
m : total number of *time instants* that replenishments occur,
τ_j : time instant when the jth replenishment occurs, $j = 1, ..., m$,
T_i : length of replenishment interval for item i,
T : length of the time horizon,
V_0 : fixed storage capacity of the warehouse,
h_i : unit cost of holding inventory for one replenishment period for item i,
p_i : unit penalty cost of item i for demand backordered for one replenishment period,

Demands

$d_{i,t}$: demand of item i during time $(t, t+1)$, a random variable, where t is the index of the time period, $t = 0, 1, ..., T$,
$D_{i,w}$: cumulative demand of item i over w time periods, where $D_{i,w} = \sum_{s=t+1}^{t+w} d_{i,s}$,

State and decision variables

$I_{i,j}$: inventory level of item i at time τ_j before replenishment, where $I_{i,1}$ are given,
$Q_{i,j}$: replenishment quantity of item i at time τ_j.

We define $R_j = \{\rho_{1,j}, ..., \rho_{k_j,j}\}$ as the set of items replenished at time τ_j, where k_j $(\leq l)$ is the number of items replenished at τ_j and $\rho_{i,j}$ denotes the index of the ith item. We also define $Q_j = (Q_{\rho_{1,j}}, ..., Q_{\rho_{k_j,j}})$ as a vector of decision variables, where $Q_{i,j}$, is the replenishment quantity of item i at time τ_j, where $i \in R_j$.

Since we allow different replenishment schedule for each item (see Figure 3.1), it is necessary to know the current inventory level of each item to check the capacity

constraint when new shipments from manufacturers occur at the warehouse. Thus, determining the replenishment size of an item depends on the current inventory level of all other items as well as the warehouse capacity. We notice in Figure 3.1 that the replenishment intervals overlap since a replenishment for one item can occur in the middle of the replenishment interval for other items.

We first define $S_j = (I_{1,j}, ..., I_{l,j})$ as the state of the system at time τ_j or in *stage* j. For a given state S_j, the current inventory level of the system in stage j is $I_j = \sum_{i=1}^{l} I_{i,j}$, which can be expressed recursively as follows:

$$I_j = I_{j-1} + \sum_{i \in R_{j-1}} Q_{i,j-1} - \sum_{i=1}^{l} D_{i,w_j} \tag{3.1}$$

where $w_j = \tau_j - \tau_{j-1}$. We also determine the current inventory levels of individual items recursively using the state in the previous stage:

$$I_{i,j} = \begin{cases} I_{i,j-1} + Q_{i,j-1} - D_{i,w_j}, & if \;\; i \in R_{j-1} \\ I_{i,j-1} - D_{i,w_j}, & otherwise \end{cases} \tag{3.2}$$

We define $g_j(S_j, Q_j)$ as the expected holding and penalty costs of the system in stage j, given initial inventories of $S_j = (I_{1,j}, ..., I_{l,j})$ and replenishment sizes of $Q_j = (Q_{\rho_{1,j}}, ..., Q_{\rho_{k_j,j}})$. The detailed expression of $g_j(S_j, Q_j)$ will be shown later. Denote the expected holding and penalty costs for stage j through m by $G_j(S_j, Q_j)$ and let $G_j^*(S_j) = \min_{Q_{i,j} \;,\; i \in R_j} G_j(S_j, Q_j)$ be the minimum expected costs in stage j through m, given the system starts stage j in state $S_j = (I_{1,j}, ..., I_{l,j})$ and is subject to the capacity constraint, $\sum_{i=1}^{l} I_{i,j} + \sum_{i \in R_j} Q_{i,j} \leq V_0$. We then state the recursive

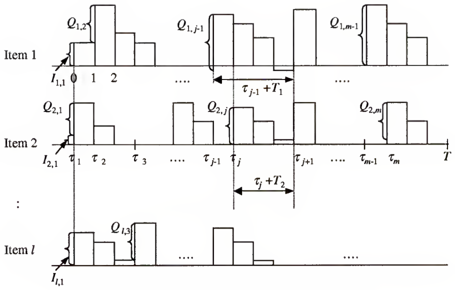


Figure 3.1: A multi-item inventory system.

equations whose solution provides an optimal policy for the problem.

$$\begin{aligned} G_j(S_j, Q_j) &= g_j(S_j, Q_j) + E\left[G^*_{j+1}(S_{j+1})\right] \\ &= g_j(S_j, Q_j) + \int_0^\infty \cdots \int_0^\infty \left[G^*_{j+1}(S_{j+1})\right] f_{1,j}(x_1) \cdots f_{l,j}(x_l) dx_1 \cdots dx_l \end{aligned} \tag{3.3}$$

where $f_{i,j}$ is the probability density function of a random demand, $D_{i,j}$.

We eventually want to find the optimal cost for the entire planning horizon, $G^*_1(I_1)$, which can be calculated recursively by solving the following equation, for $j = 2, ..., m$,

$$
\begin{aligned}
G_j^*(S_j) &= \min_{Q_{i,j}:\sum_{i=1}^{l} I_{i,j}+\sum_{i\in R_j} Q_{i,j}\le V_0} \left[g_j(S_j,Q_j) + \int_0^\infty \cdots \right. \\
&\left. \int_0^\infty \left[G_{j+1}^*(S_{j+1})\right] f_{1,j}(x_1)\cdots f_{l,j}(x_l)dx_1\cdots dx_l \right] \qquad (3.4)
\end{aligned}
$$

We now express the single-period expected cost in any stage j as follows:

$$
\begin{aligned}
g_j(S_j,Q_j) &= \sum_{i\in R_j} E\left[h_i(I_{i,j}+Q_{i,j}-D_{i,T_j})^+ + p_i(D_{i,T_j}-I_{i,j}-Q_{i,j})^+\right] \\
&= \sum_{i\in R_j}\int_0^\infty \left[h_i(I_{i,j}+Q_{i,j}-x_i)^+ + p_i(x_i-I_{i,j}-Q_{i,j})^+\right] f_{i,j}(x_i)dx_i \\
&= \sum_{i\in R_j}\left[h_i\int_{x_i=0}^{I_{i,j}+Q_{i,j}} (I_{i,j}+Q_{i,j}-x_i) f_{i,j}(x_i)dx_i \right. \\
&\qquad \left. + p\int_{x_i=I_{i,j}+Q_{i,j}}^\infty (x_i-I_{i,j}-Q_{i,j}) f_{i,j}(x_i)dx_i\right] \qquad (3.5)
\end{aligned}
$$

where $f_{i,j}$ is the probability density function of the demand for item i in period j.

Note that this expected cost only includes those items that are replenished at time τ_j. The reason for this can be explained as follows: for simplicity, we only consider the first two items in Figure 3.1. Since item 2 is scheduled to be replenished at time τ_j, we consider one replenishment interval of the item, which ranges between τ_j and $\tau_j + T_2$. Then the total cost in this stage is the sum of the cost for item 1 incurred between τ_j and $\tau_j + T_2$ (say, cost 1) and the cost for item 2 incurred between τ_j and $\tau_j + T_2$ (say, cost 2). Cost 1, however, is not affected by the current decision variable, $Q_{2,j}$, whereas the corresponding item provides information about its current inventory level to the system in order to check the capacity constraint, $\sum_{i=1}^{2} I_{i,\tau_j} + Q_{2,j} \le V_0$. That is, the information about the current inventory level of

item 1 is embedded in the capacity constraint. How about the cost related to item 1 (cost 1)? As we mentioned before, cost 1 is not affected by the decision variable, $Q_{2,j}$. Thus, when we determine the optimal policy, the final decision is not affected by whether we include the cost in the objective function or not.

We now look at stage $(j-1)$, which runs from τ_{j-1} to $\tau_{j-1}+T_1$. In this stage, the decision variable is $Q_{1,j-1}$. Note that the cost for item 1 incurred between τ_{j-1} to $\tau_{j-1}+T_1$ includes cost 1. That is, cost 1 is double counted in periods j and $(j-1)$. This also happens to item 2 (cost 2 is counted twice) and the cost is not affected by $Q_{1,j-1}$. To avoid this double counting, we remove one from where it is not necessary to be counted (for instance, cost 1 in stage j and cost 2 in stage $(j-1)$). Once we remove it, the total expected holding and penalty cost in stage j only includes the cost for item 2 as in (3.5). For any stage, these double-counted costs can be removed in this fashion, and finally, the expected cost for the entire system is well defined. For more than two items, we can easily extend this to express the corresponding expected cost by using the aforementioned method.

3.3 Solution Approaches

We next discuss solution approaches to the dynamic recursive equation (3.4), which is subject to the capacity constraint. In the previous chapter we established a solution approach for minimizing expected cost in the case of equal replenishment intervals (periodic-review case). We also found that, in any stage j, the optimal replenishment quantity was independent of the future cost incurred in that stage. This argument is generally not true anymore in the case of unequal replenishment

intervals, except in the final stage of the planning horizon (stage m in our case). To show this, we first start with a simple case of a two-item, two-period inventory system, assuming that each replenishment occurs in time periods 1 and 2.

Two-item, Two-period Case

Consider the warehouse inventory system with $l = 2$ items and $m = 2$ periods where item 1 is replenished at time $\tau_1 = 1$ and item 2 is replenished at time $\tau_2 = 2$. See Figure 3.2. We use same notation that we defined earlier in this chapter, except for defining Q_j as the replenishment quantity in stage j, for simplicity.

Given that $I_{1,1}$ and $I_{2,1}$ are known, the inventory level in period (or stage) 2 can be expressed as follows:

$$I_{1,2} = I_{1,1} + Q_1 - d_{1,1}, \qquad I_{2,2} = I_{2,1} - d_{2,1}$$

Note that, from Figure 3.2, the expected cost in period 2, $g_2(I_{1,2}, I_{2,2}, Q_2)$, only includes the expected cost of item 2 incurred in period 2, whereas the expected cost in period 1, $g_1(I_{1,1}, I_{2,1}, Q_1)$, consists of the expected cost of item 1 incurred in one full replenishment period from time 1 to T as well as the expected cost of item 2 incurred in period 1. Then the expected cost for period 2 through the end of planning horizon, $G_2(I_{1,2}, I_{2,2}, Q_2)$, is identical to $g_2(I_{1,2}, I_{2,2}, Q_2)$ and can be stated as follows:

$$\begin{aligned} G_2(I_{1,2}, I_{2,2}, Q_2) &= g_2(I_{1,2}, I_{2,2}, Q_2) + E[G_3^*(I_{1,2} - d_{1,2}, I_{2,2} + Q_2 - d_{2,2})] \\ &= E_{d_{2,2}}[h_2(I_{2,2} + Q_2 - d_{2,2})^+ + p_2(d_{2,2} - I_{2,2} - Q_2)^+] \quad (3.6) \end{aligned}$$

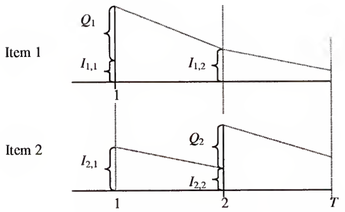


Figure 3.2: A two-item, two period inventory system.

(assuming that $G_3^* = 0$). Applying the same method as in the periodic-review case, we can solve (3.6) to determine Q_2^*, which becomes a function of Q_1, $d_{1,1}$ and $d_{2,1}$, as follows:

$$\begin{aligned}
Q_2^*(Q_1, d_{1,1}, d_{2,1}) &= F_{2,2}^{-1}\left(\frac{p_2 - \mu_2^*(Q_1, d_{1,1})}{h_2 + p_2}\right) - I_{2,2} \\
&= F_{2,2}^{-1}\left(\frac{p_2 - \mu_2^*(Q_1, d_{1,1})}{h_2 + p_2}\right) - (I_{2,1} - d_{2,1}) \\
\mu_2^*(Q_1, d_{1,1}) &= arg \min_{\mu_2 \geq 0}\left[F_{2,2}^{-1}\left(\frac{p_2 - \mu_2}{h_2 + p_2}\right) + I_{1,2} \leq V_0\right] \\
&= arg \min_{\mu_2 \geq 0}\left[F_{2,2}^{-1}\left(\frac{p_2 - \mu_2}{h_2 + p_2}\right) + I_{1,1} + Q_1 - d_{1,1} \leq V_0\right]
\end{aligned}$$

or simply

$$Q_2^*(Q_1, d_{1,1}, d_{2,1}) = \min\left[F_{2,2}^{-1}\left(\frac{p_2}{h_2 + p_2}\right), V_0 - (I_{1,1} + Q_1 - d_{1,1})\right] - I_{2,1} + d_{2,1} \tag{3.7}$$

where $F_{i,j}$ is the *cdf* of $d_{i,j}$. Note that μ^* here has an extra term, $I_{1,2}$, compared to that in the periodic-review case in the previous chapter, (2.18). $I_{1,2}$ is the current inventory level of item 1 which is not replenished in period 2. Since all items are

replenished at the same time in the periodic-review case, we do not have this extra term in that case. Even though we do not replenish item 1 in period 2, we still need information about inventory levels of this item to check the capacity constraint. The expected cost function for period 1 through the end of planning horizon is then

$$
\begin{aligned}
G_1(I_{1,1}, I_{2,1}, Q_1) &= g_1(I_{1,1}, I_{2,1}, Q_1) + E_{d_{1,1}}\left[G_2^*(I_{1,1} + Q_1 - d_{1,1}, I_{2,1} - d_{2,1})\right] \\
&= E_{D_1}\left[h_1(I_{1,1} + Q_1 - D_1)^+ + p_1(D_1 - I_{1,1} - Q_1)^+\right] \\
&\quad + E_{d_{2,1}}\left[h_2(I_{2,1} - d_{2,1})^+ + p_2(d_{2,1} - I_{2,1})^+\right] \\
&\quad + E_{d_{1,1}}\left[E_{d_{2,1}}\left[E_{d_{2,2}}\left[h_2\left(I_{2,1} + d_{2,1} + Q_2^*(Q_1, d_{1,1}, d_{2,1}) - d_{2,2}\right)^+\right.\right.\right. \\
&\qquad \left.\left.\left. + \; p_2\left(d_{2,2} - I_{2,1} - d_{2,1} - Q_2^*(Q_1, d_{1,1}, d_{2,1})\right)^+\right]\right]\right] \quad (3.8)
\end{aligned}
$$

where $D_1 = d_{1,1} + d_{1,2}$. The first term in the right hand side in (3.8) is the expected cost of item 1 incurred in both periods 1 and 2, and the second term is the expected cost of item 2 incurred in period 1. We now state the following proposition.

Proposition 3.1 *The expected holding and penalty cost in period 1, $G_1(I_{1,1}, I_{2,1}, Q_1)$, with under limited capacity is a convex function of the replenishment size, Q_1.*

Proof : Let $L_1 = G_1(I_{1,1}, I_{2,1}, Q_1) + \mu_1(I_{1,1} + I_{2,1} + Q_1 - V_0)$. Taking the first and second partial derivatives on L_1 with respect to Q_1, we get the following:

$$
\begin{aligned}
\frac{\partial L_1}{\partial Q_1} &= \frac{\partial}{\partial Q_1}\left[G_1(I_{1,1}, I_{2,1}, Q_1) + \mu_1(I_{1,1} + I_{1,2} + Q_1 - V_0)\right] \\
&= \frac{\partial}{\partial Q_1}\left[E_{D_1}\left[h_1(I_{1,1} + Q_1 - D_1)^+ + p_1(D_1 - I_{1,1} - Q_1)^+\right]\right] \\
&\quad + \frac{\partial}{\partial Q_1} E_{d_{1,1}} E_{d_{2,1}}\left[E_{d_{2,2}}\left[h_2\left(I_{2,1} - d_{2,1} + Q_2^*(Q_1, d_{1,1}, d_{2,1}) - d_{2,2}\right)^+\right.\right. \\
&\qquad \left.\left. + p_2\left(d_{2,2} - I_{2,1} + d_{2,1} - Q_2^*(Q_1, d_{1,1}, d_{2,1})\right)^+\right]\right] + \mu_1
\end{aligned}
$$

$$
\begin{aligned}
&= (h_1+p_1)F_1(I_{1,1}+Q_1)-p_1+\mu_1 \\
&\quad +\frac{\partial}{\partial Q_1}\int_{d_{1,1}}\int_{d_{2,2}}\left[h_2\left(A(Q_1,d_{1,1})-d_{2,2}\right)^+\right. \\
&\qquad\qquad \left.+p_2\left(d_{2,2}-A(Q_1,d_{1,1})\right)^+\right] f_{2,2}f_{1,1}\mathrm{d}d_{2,2}\mathrm{d}d_{1,1} \\
&= (h_1+p_1)F_1(I_{1,1}+Q_1)-p_1+\mu_1 \\
&\quad +\frac{\partial}{\partial Q_1}\int_{d_{1,1}}\left[\int_{d_{2,2}=0}^{A(Q_1,d_{1,1})} h_2\left(A(Q_1,d_{1,1})-d_{2,2}\right)f_{2,2}\mathrm{d}d_{2,2}\right. \\
&\qquad \left.+\int_{d_{2,2}=A(Q_1,d_{1,1})}^{\infty} p_2\left(d_{2,2}-A(Q_1,d_{1,1})\right)f_{2,2}\mathrm{d}d_{2,2}\right] f_{1,1}\mathrm{d}d_{1,1} \qquad (3.9) \\
&= (h_1+p_1)F_1(I_{1,1}+Q_1)-p_1+\mu_1 \\
&\quad +\frac{\partial}{\partial Q_1}\int_{d_{1,1}=a(Q_1)}^{\infty}\left[\int_{d_{2,2}=0}^{F_{2,2}^{-1}} h_2\left(F_{2,2}^{-1}-d_{2,2}\right)f_{2,2}\mathrm{d}d_{2,2}\right. \\
&\qquad \left.+\int_{d_{2,2}=F_{2,2}^{-1}}^{\infty} p_2\left(d_{2,2}-F_{2,2}^{-1}\right)f_{2,2}\mathrm{d}d_{2,2}\right] f_{1,1}\mathrm{d}d_{1,1} \\
&\quad +\frac{\partial}{\partial Q_1}\int_{d_{1,1}=0}^{a(Q_1)}\left[\int_{d_{2,2}=0}^{b(Q_1)} h_2\left(b(Q_1)-d_{2,2}\right)f_{2,2}\mathrm{d}d_{2,2}\right. \\
&\qquad \left.+\int_{d_{2,2}=b(Q_1)}^{\infty} p_2\left(d_{2,2}-b(Q_1)\right)f_{2,2}\mathrm{d}d_{2,2}\right] f_{1,1}\mathrm{d}d_{1,1} \qquad (3.10)
\end{aligned}
$$

where F_1 is *cdf* of D_1, $f_{i,j}$ is *pdf* of $d_{i,j}$, $A(Q_1,d_{1,1})=\min\left[F_{2,2}^{-1}\left(\frac{p_2}{h_2+p_2}\right),V_0-I_{1,2}\right]=\min\left[F_{2,2}^{-1}\left(\frac{p_2}{h_2+p_2}\right),V_0-I_{1,1}-Q_1+d_{1,1}\right]$ from (3.7), $a(Q_1)=F_{2,2}^{-1}-V_0+I_{1,1}+Q_1$ and $b(Q_1)=V_0-I_{1,1}-Q_1+d_{1,1}$. $a(Q_1)$ and $b(Q_1)$ in (3.10) are obtained as follows: From (3.9), we get

$$
A(Q_1,d_{1,1})=\begin{cases} F_{2,2}^{-1}\left(\frac{p_2}{h_2+p_2}\right), & if \;\; F_{2,2}^{-1}\left(\frac{p_2}{h_2+p_2}\right)\le V_0-I_{1,2} \\ V_0-I_{1,2}, & otherwise \end{cases}
$$

and then

$$
\begin{aligned}
& F_{2,2}^{-1}\left(\frac{p_2}{h_2+p_2}\right) \leq V_0 - I_{1,2} \\
\Longleftrightarrow \quad & F_{2,2}^{-1}\left(\frac{p_2}{h_2+p_2}\right) \leq V_0 - (I_{1,1} + Q_1 - d_{1,1}) \\
\Longleftrightarrow \quad & d_{1,1} \geq F_{2,2}^{-1}\left(\frac{p_2}{h_2+p_2}\right) - V_0 + I_{1,1} + Q_1 = a(Q_1) \\
\text{and} \quad & b(Q_1) = V_0 - I_{1,2} = V_0 - I_{1,1} - Q_1 + d_{1,1}
\end{aligned}
$$

Returning back to (3.10), we have

$$
\begin{aligned}
\frac{\partial L_1}{\partial Q_1} = \; & (h_1+p_1)F_1(I_{1,1}+Q_1) - p_1 + \mu_1 \\
& - \left[\int_{d_{2,2}=0}^{F_{2,2}^{-1}} h_2\left(F_{2,2}^{-1} - d_{2,2}\right) f_{2,2} \mathrm{d}d_{2,2} \right. \\
& \qquad \left. + \int_{d_{2,2}=F_{2,2}^{-1}}^{\infty} p_2\left(d_{2,2} - F_{2,2}^{-1}\right) f_{2,2} \mathrm{d}d_{2,2}\right] f_{1,1}(a(Q_1)) \\
& + \left[\int_{d_{2,2}=0}^{F_{2,2}^{-1}} h_2\left(F_{2,2}^{-1} - d_{2,2}\right) f_{2,2} \mathrm{d}d_{2,2} \right. \\
& \qquad \left. + \int_{d_{2,2}=F_{2,2}^{-1}}^{\infty} p_2\left(d_{2,2} - F_{2,2}^{-1}\right) f_{2,2} \mathrm{d}d_{2,2}\right] f_{1,1}(a(Q_1)) \\
& + \int_{d_{1,1}=0}^{a(Q_1)} \frac{\partial}{\partial Q_1}\left[\int_{d_{2,2}=0}^{b(Q_1)} h_2\left(b(Q_1) - d_{2,2}\right) f_{2,2} \mathrm{d}d_{2,2} \right. \\
& \qquad \left. + \int_{d_{2,2}=b(Q_1)}^{\infty} p_2\left(d_{2,2} - b(Q_1)\right) f_{2,2} \mathrm{d}d_{2,2}\right] f_{1,1} \mathrm{d}d_{1,1} \\
= \; & (h_1+p_1)F_1(I_{1,1}+Q_1) - p_1 + \mu_1 \\
& + \int_{d_{1,1}=0}^{a(Q_1)} \frac{\partial}{\partial Q_1}\left[\int_{d_{2,2}=0}^{b(Q_1)} h_2\left(b(Q_1) - d_{2,2}\right) f_{2,2} \mathrm{d}d_{2,2} \right. \\
& \qquad \left. + \int_{d_{2,2}=b(Q_1)}^{\infty} p_2\left(d_{2,2} - b(Q_1)\right) f_{2,2} dd_{2,2}\right] f_{1,1} \mathrm{d}d_{1,1} \\
= \; & (h_1+p_1)F_1(I_{1,1}+Q_1) - p_1 + \mu_1 \\
& + \int_{d_{1,1}=0}^{a(Q_1)} \left[\int_{d_{2,2}=0}^{b(Q_1)} -h_2 f_{2,2} \mathrm{d}d_{2,2} + \int_{d_{2,2}=b(Q_1)}^{\infty} p_2 f_{2,2} \mathrm{d}d_{2,2}\right] f_{1,1} \mathrm{d}d_{1,1}
\end{aligned}
$$

$$
\begin{aligned}
&= (h_1+p_1)F_1(I_{1,1}+Q_1)-p_1+\mu_1 \\
&\quad+\int_{d_{1,1}=0}^{a(Q_1)}\left[-h_2\int_{d_{2,2}=0}^{b(Q_1)} f_{2,2}\mathrm{d}d_{2,2}-p_2\int_{d_{2,2}=0}^{b(Q_1)} f_{2,2}\mathrm{d}d_{2,2}+p_2\right] f_{1,1}\mathrm{d}d_{1,1} \\
&= (h_1+p_1)F_1(I_{1,1}+Q_1)-p_1+\mu_1 \\
&\quad+p_2F_{1,1}(a(Q_1))-(h_2+p_2)\int_{d_{1,1}=0}^{a(Q_1)} F_{2,2}(b(Q_1))f_{1,1}\mathrm{d}d_{1,1} \qquad (3.11) \\
\frac{\partial^2}{\partial {Q_1}^2} &= (h_1+p_1)f_{1,1}(I_{1,1}+Q_1) \\
&\quad+(h_2+p_2)\left[\frac{p_2}{h_2+p_2}\right] f_{1,1}(a(Q_1)) \\
&\quad+(h_2+p_2)\int_{d_{1,1}=0}^{a(Q_1)} f_{2,2}(b(Q_1))f_{1,1}(a(Q_1))\mathrm{d}d_{1,1} \geq 0 \quad \square
\end{aligned}
$$

The result in Proposition 3.1 shows that the two-item, two-period inventory model has an unique optimal replenishment policy and the solution satisfying the first-order conditions is optimal. This solution approach would work in rather simple cases with small number of items or periods. When the number of items or periods increase, the number of terms to be evaluated (e.g. those in (3.10)) increases rapidly and it would be impractical to solve the problem using the optimal solution approach. This leads us to propose heuristics to determine replenishment policies.

3.4 Heuristic Algorithms

For clarity of presentation, we first consider two-item inventory systems, assuming that only one item is replenished in a period. Two distinct replenishment schedules are used to formulate heuristics and we then extend those heuristics to more general cases.

3.4.1 Heuristics for a Basic Replenishment Schedule

The first replenishment schedule we consider is where item 1 is replenished in every odd period and item 2 is replenished in every even period until the end of the time horizon (see Figure 3.3). We call this replenishment schedule Schedule I.

The first heuristic (Heuristic A) is motivated from Proposition 2.6 in the previous chapter where we dealt a problem with equal replenishment intervals. Referring to (2.17) and (2.18), we can see that when the capacity constraint is tight, the overall capacity of the warehouse in any period j, V_0, is divided into two terms as follows:

$$F_{1,j}^{-1}\left(\frac{p_1-\mu_j}{h_1+p_1}\right)+F_{2,j}^{-1}\left(\frac{p_2-\mu_j}{h_2+p_2}\right)=V_0, \quad \mu_j>0 \tag{3.12}$$

Note that each of terms on the left hand side in (3.12) is just a function of unit holding and penalty costs and the demand distribution of items 1 and 2, respectively, and those values are given. The first term is related to item 1 and the second term is related to item 2. When the capacity constraint is not tight, we have

$$F_{1,j}^{-1}\left(\frac{p_1-\mu_j}{h_1+p_1}\right)+F_{2,j}^{-1}\left(\frac{p_2-\mu_j}{h_2+p_2}\right)<V_0, \quad \mu_j=0 \tag{3.13}$$

We now apply these results to the case with unequal replenishment intervals. We assume that the demand of each item is identically distributed over periods, and thus we can drop subscript j. We project the replenishment schedule of item 1 and item 2 in Figure 3.3 so that both items have the same replenishment schedule (It is possible only in this particular case because the length of replenishment intervals for each item is identical. We consider more general cases later.) We then consider the

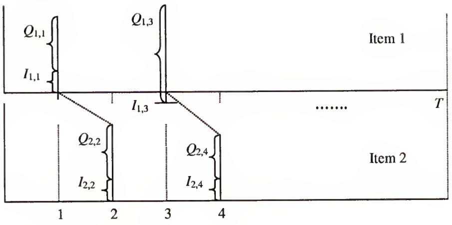


Figure 3.3: An inventory system with unequal replenishment intervals (Schedule I).

first terms in both (3.12) and (3.13) as the capacity of item 1, V_1, and the second terms as the capacity of item 2, V_2, over the entire planning horizon. That is, we split the overall capacity, V_0, into two individual capacities. We then determine V_1 and V_2 by solving the following:

$$
\begin{aligned}
&\textit{Find} && \mu^* = \min\{\mu \geq 0\} \\
&s.t. && V_1 + V_2 \leq V_0 \\
& && V_1 = F_1^{-1}\left(\frac{p_1 - \mu}{h_1 + p_1}\right) \\
& && V_2 = F_2^{-1}\left(\frac{p_2 - \mu}{h_2 + p_2}\right)
\end{aligned}
\tag{3.14}
$$

Note that F_i is the cumulative distribution function of item i for two periods in this case. Once we determine those individual capacities for items, we return back to our original problem and assume that each item has a separate capacity (see Figure 3.4) The replenishment quantity of the item replenished in period j is limited by

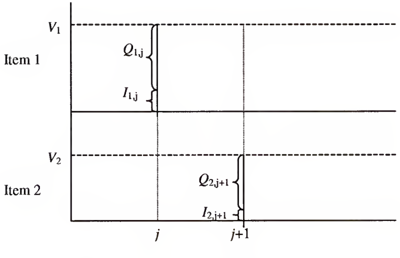


Figure 3.4: An inventory policy of Heuristic A.

its own capacity. That is, when we determine the replenishment quantity of an item in period j, we only need to consider the corresponding capacity and current inventory level of that item. Once each item has its own capacity, we have two separate periodic-review inventory systems and thus we can determine the respective optimal replenishment quantity (for the separated problem) of each item separately. Note that individual capacities depend on demands and unit costs and thus reflect this information when determining capacities. This heuristic always replenishes item i up to V_i, $i = 1, 2$. Referring to (3.13), we can see that the size of V_i depends on whether the overall capacity constraint is tight ($\mu > 0$) or not ($\mu = 0$). When demand rates are not high, compared to the warehouse capacity, the overall capacity constraint is not tight and the sum of two individual capacities should be less than the total capacity, V_0. Then the heuristic has a replenishment policy which replenishes item i up to the unconstrained optimal inventory level or $V_i = F_i^{-1}\left(\frac{p_i}{h_i+p_i}\right)$, assuming that

item i is replenished in period j, and the policy becomes optimal for the constraint case as well. When the constraint is tight, Heuristic A replenishes item i up to $V_i = F_i^{-1}\left(\frac{p_i-\mu^*}{h_i+p_i}\right)$ where $V_1 + V_2 = V_0$. One possible disadvantage of this heuristic is that the warehouse might have unused capacity for one item while the other item needs a greater replenishment quantity than its own capacity to meet demand. In most cases, however, this situation would be resolved by the dependency of capacity on demands because when one item has a higher demand the corresponding capacity would be increased as well. Therefore, overall, this capacity-separate scheme would be plausible even though, in some individual periods, the aforementioned disadvantages might occur. Finally, each item is replenished up to its own capacity.

The second heuristic (Heuristic B) is similar to the first one in the sense that we use a separate capacity for each item. Instead of using fixed capacities, V_1 and V_2, this heuristic adjusts those capacities according to the availability of any unused capacity. This adjustment is based on the following: Assuming that item 1 is replenished in period j and item 2 is not, the inventory level of item 2 is less than its own capacity, V_2, because the inventory level has been decreasing ever since the last replenishment of item 2 (see Figure 3.5). Thus, there is an unused space for item 2 at the beginning of period j and we want to allocate at least a part of this unused capacity to item 1 so that item 1 has more available space to accommodate its demand. Then a question arises immediatelyhow much we can increase the capacity of item 1. Basically, we increase it as much as the average demand of item 1 incurred between time j and $(j+1)$ so that the capacity of item 1 eventually is reduced back to V_1 at the beginning of period $(j+1)$, and the capacity of item 2 then decreases as much as its average

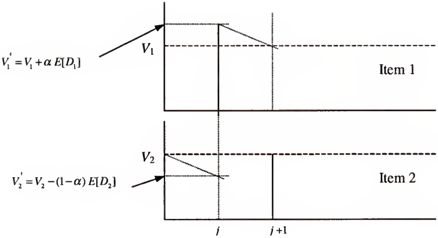


Figure 3.5: An inventory policy of Heuristic B.

demand for one period as long as the overall capacity in period j does not exceed V_0. As a result, the newly adjusted capacity can then be determined by solving the following equations

$$\begin{aligned} V_1 + \alpha E[D_1] + V_2 - (1-\alpha)E[D_2] &= V_0 \\ \alpha E[D_1] - (1-\alpha)E[D_2] &= 0 \end{aligned} \tag{3.15}$$

where α is a adjustment factor and solving (3.15) results

$$V_1' = V_1 + \alpha E[D_1]$$

$$\begin{aligned} V_2' &= V_2 - (1-\alpha)E[D_2] \\ \alpha &= \frac{E[D_2]}{E[D_1] + E[D_2]} \end{aligned}$$

If item 2 is replenished in period j, we get

$$V_1 - (1-\alpha)E[D_1] + V_2 + \alpha E[D_2] \;=\; V_0$$

and

$$\begin{aligned} V_1' &= V_1 - (1-\alpha)E[D_1] \\ V_2' &= V_2 + \alpha E[D_2] \\ \alpha &= \frac{E[D_1]}{E[D_1] + E[D_2]} \end{aligned}$$

We apply this scheme in any period by increasing the capacity of the item replenished in that period and by decreasing that of the other. Once the capacity is adjusted, an individual item is replenished up to its inventory level, which is the minimum of its adjusted capacity (V_k), the unconstrained optimal inventory level after replenishment ($F_i^{-1}\left(\frac{p_i}{h_i+p_i}\right)$), or the total available capacity ($V_0 - I_{k,j}$), assuming that item i is planned to be replenished in period j, where $k = 1, 2, k \neq i$. The adjustment of capacity wouldn't hurt the overall performance of Heuristic B because it only allocates any available space to where it needs and does not force item 2 to reduce its physical inventory level due to the capacity reduction. Even though item 1 has a capacity increment, it adjusts its inventory level only to a direction to improve the overall performance. Therefore, this heuristic would improve the overall performance from Heuristic A.

In the last heuristic (Heuristic C), we consider the total available capacity, $V_0 - I_{2,j}$, in the system at the time instant of replenishment instead of using separate capacity for each item. Referring to Figure 3.6, item i is replenished up to its total

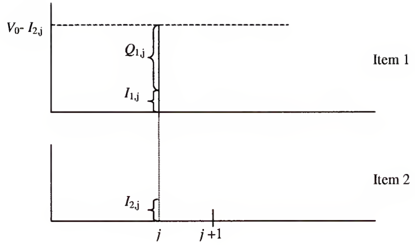


Figure 3.6: An inventory policy of Heuristic C.

available capacity, or the unconstrained optimal inventory level after replenishment, $F_i^{-1}\left(\frac{p_i}{h_i+p_i}\right)$, whichever is smaller. Thus, heuristic C always maintains the greatest capacity for replenishment among three. Note that when the capacity constraint is not tight, all three heuristics have the same replenishment policy which replenishes up to $F_i^{-1}\left(\frac{p_i}{h_i+p_i}\right)$. We summarize these three heuristics in the following.

Heuristic A

- Compute the separate capacity of each item by solving the following:

$$
\begin{aligned}
Find \quad & \mu^* = \min\{\mu \geq 0\} \\
s.t. \quad & V_1 + V_2 \leq V_0 \\
& V_1 = F_1^{-1}\left(\frac{p_1 - \mu}{h_1 + p_1}\right) \\
& V_2 = F_2^{-1}\left(\frac{p_2 - \mu}{h_2 + p_2}\right)
\end{aligned}
$$

- Determine the replenishment quantity

$$Q_{i,j} = V_i - I_{i,j} \ , \quad i = 1, 2$$

Heuristic B

- Compute the separate capacity of each item as for Heuristic A.
- Adjust the capacities of individual items based on their demands as follows:

  · If item 1 is replenished in period j,

$$\begin{aligned} V_1' &= V_1 + \alpha E[D_1] \\ V_2' &= V_2 - (1-\alpha)E[D_2] \\ \alpha &= \frac{E[D_2]}{E[D_1] + E[D_2]} \end{aligned}$$

  · If item 2 is replenished in period j,

$$\begin{aligned} V_1' &= V_1 - (1-\alpha)E[D_1] \\ V_2' &= V_2 + \alpha E[D_2] \\ \alpha &= \frac{E[D_1]}{E[D_1] + E[D_2]} \end{aligned}$$

- Determine the replenishment quantity

$$Q_{i,j} = \min\left[F_i^{-1}\left(\frac{p_i}{h_i + p_i}\right), \ V_i', \ V_0 - I_{k,j}\right] - I_{i,j} \ , \quad i, k = 1, 2, k \neq i$$

Heuristic C

- Assuming that item i is replenished in period j, determine the available capacity, $V_0 - I_{k,j}, \quad i, k = 1, 2, k \neq i$ for item i.

- Determine the replenishment quantity

$$Q_{i,j} = \min\left[F_i^{-1}\left(\frac{p_i}{h_i+p_i}\right),\ V_0 - I_{k,j}\right] - I_{i,j}\ ,\quad i,k = 1,2, k \neq i$$

3.4.2 Heuristics for An Alternative Replenishment Schedule

We now consider a more general replenishment schedule (Schedule II). Let item 1 be replenished in periods 1, 3, 5, 7, ... and item 2 be replenished in periods 4, 8, 12, 16, ... See Figure 3.7. To determine individual capacities of items, we first project the replenishment schedule of item 2 on that of item 1 as in the previous case. As we can notice, the length of replenishment interval of each item is not equal. We then find the least common multiple of individual interval lengths, which is four in this case (we call this the length of one replenishment cycle). Since item 1 has two replenishments in one replenishment cycle while item 2 has one, the expected cost for one replenishment cycle can be rewritten as follows:

$$E[2h_1(y_1-d_1)^+ + 2p_1(d_1-y_1)^+ + h_2(y_2-d_2)^+ + p_2(d_2-y_2)^+]$$

where y_i is replenish up to level and is equivalent to the inventory level of item i after replenishment. By minimizing the expected cost with respect to y_i under the warehouse constraint, the following is obvious from the previous derivation

$$\begin{aligned}
y_1^* &= F_1^{-1}\left(\frac{2p_1-\mu^*}{2(h_1+p_1)}\right), \\
y_2^* &= F_2^{-1}\left(\frac{p_2-\mu^*}{h_2+p_2,}\right) \\
\mu^* &= arg\ \min_{\mu\geq 0}\left[F_1^{-1}\left(\frac{2p_1-\mu}{2(h_1+p_1)}\right) + F_2^{-1}\left(\frac{p_2-\mu}{h_2+p_2,}\right) \leq V_0\right]
\end{aligned}$$

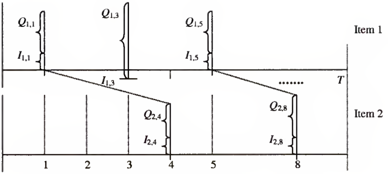


Figure 3.7: An inventory system with unequal replenishment intervals (Schedule II).

Using the same approach as in the previous section, we determine individual capacity of each item by solving the following

$$\begin{aligned} Find \quad & \mu^* = \min\{\mu \geq 0\} \\ s.t. \quad & V_1 + V_2 \leq V_0 \\ & V_1 = F_1^{-1}\left(\frac{2p_1 - \mu}{2(h_1 + p_1)}\right) \\ & V_2 = F_2^{-1}\left(\frac{p_2 - \mu}{h_2 + p_2}\right) \end{aligned}$$

and Heuristic A has a replenishment policy which replenishes item i up to V_i.

Heuristic C remains the same as in the previous case. In other words, the heuristic replenishes item i up to its total available capacity, or the unconstrained optimal inventory level after replenishment, $F_i^{-1}\left(\frac{p_i}{h_i+p_i}\right)$, whichever is smaller.

For Heuristic B, we first take a look at Figure 3.7. In periods j =1, 5, 9,..., we can see that no replenishment occurs in the following periods (periods 2,6,10,...). Hence, we increase the capacity of item 1 as much as the average demand of item 1 in the current and following periods (periods j and $j+1$), so that the adjusted capacity

is eventually reduced back to its original capacity after two periods. For item 2, we decrease its capacity as much as the average demand between its last replenishment period (periods 0, 4, 8, ...) and the current period, which is equal to $E[D_2]$. We then can determine the adjusted capacity of each item as follows:

$$\begin{aligned} V_1 + \alpha E[2D_1] + V_2 - (1-\alpha)E[D_2] &= V_0 \\ \alpha E[2D_1] - (1-\alpha)E[D_2] &= 0 \end{aligned} \tag{3.16}$$

where α is a adjustment factor and solving (3.16) results

$$\begin{aligned} V_1' &= V_1 + 2\alpha E[D_1] \\ V_2' &= V_2 - (1-\alpha)E[D_2] \\ \alpha &= \frac{E[D_2]}{2E[D_1] + E[D_2]} \end{aligned}$$

In periods 3, 7, 11,..., a replenishment occurs immediately in the following periods 4, 8, 12,... and item 2 was not replenished for the three preceding periods. Therefore, we increase the capacity of item 1 as much as $E[D_1]$ and decrease the capacity of item 2 by $3E[D_2]$.

$$\begin{aligned} & V_1 + \alpha E[D_1] + V_2 - (1-\alpha)E[3D_2] = V_0 \\ & \alpha E[D_1] - (1-\alpha)E[3D_2] = 0 \\ \Longrightarrow\quad & V_1' = V_1 + \alpha E[D_1] \\ & V_2' = V_2 - (1-\alpha)E[3D_2], \quad \alpha = \frac{3E[D_2]}{E[D_1] + 3E[D_2]} \end{aligned}$$

In periods 4, 8, 12,..., replenishment occurs immediately in the following periods 5, 9, 13,... while item 1 is not replenished for only one preceding period. We thus increase the capacity of item 2 by $E[D_2]$ and decrease the capacity of item 1 up to $E[D_2]$.

$$V_1 - (1-\alpha)E[D_1] + V_2 + \alpha E[D_2] = V_0$$

$$-(1-\alpha)E[D_1] + \alpha E[D_2] = 0$$

$$\Longrightarrow$$

$$V_1' = V_1 - (1-\alpha)E[D_1]$$

$$V_2' = V_2 + \alpha E[D_2]$$

$$\alpha = \frac{E[D_1]}{E[D_1] + E[D_2]}$$

To combine these three cases, let a_j be the number of periods between the jth and $(j+1)$th replenishment in the system. For example, in Schedule II, $a_1 = 2, a_3 = 1, a_4 = 1, a_5 = 2$, so on. Assuming that item i is not replenished in period j, we define $e_{i,j}$ as the number of preceding periods since the last replenishment of item i. For instance, $e_{2,1} = 1$, $e_{3,2} = 2$, $e_{4,1} = 1$ so on. Then we have the following:

- If item 1 is replenished in period j,

$$\begin{aligned} V_1' &= V_1 + \alpha E[a_j\ D_1] \\ V_2' &= V_2 - (1-\alpha)E[e_{2,j}\ D_2] \\ \alpha &= \frac{e_{2,j}\ E[D_2]}{a_j\ E[D_1] + e_{2,j}\ E[D_2]} \end{aligned}$$

· If item 2 is replenished in period j,

$$\begin{aligned} V_1' &= V_1 - (1-\alpha)E[e_{1,j}\ D_1] \\ V_2' &= V_2 + \alpha E[a_j\ D_2], \quad \alpha = \frac{e_{1,j}\ E[D_1]}{e_{1,j}\ E[D_1] + a_j\ E[D_2]} \end{aligned}$$

Once the adjusted capacities are determined, Heuristic B replenishes item i up to its capacity (V_i), the unconstrained optimal inventory level after replenishment ($F_i^{-1}\left(\frac{p_i}{h_i+p_i}\right)$), or total available capacity ($V_0 - I_{k,j},\ k = 1, 2, k \neq i$) whichever is smaller. We notice that this is a general version of Heuristic B for any two-item case, assuming that no simultaneous replenishment occurs in a period.

3.4.3 Heuristics for Multi-Item Cases

Based on the results from the two-item cases, we can extend these Heuristics to multi- item cases, assuming that there is no simultaneous replenishment in one period. For Heuristic A, let l be the number of items in the system and b be the least common multiple of individual interval length (or the length of one replenishment cycle). If T_i represents the replenishment interval length of item i as defined earlier in this chapter, $b_i = b/T_i$ represents the number of replenishment of item i in one cycle. Then the expected cost for one replenishment cycle is

$$\sum_{i=1}^{l} E\left[b_i\left(h_i(y_i - d_i)^+ + p_i(d_i - y_i)^+\right)\right]$$

and the following is immediate

$$y_i^* = F_i^{-1}\left(\frac{b_i p_i - \mu^*}{b_i(h_i + p_i)}\right), \quad i = 1, 2, ..., l,$$

$$\mu^* = arg \min_{\mu \geq 0} \left[\sum_{i=1}^{l} F_i^{-1} \left(\frac{b_i p_i - \mu}{b_i(h_i + p_i)} \right) \leq V_0 \right]$$

The capacity of each item can then be obtained by solving the following:

$$\begin{aligned} Find \quad & \mu^* = \min\{\mu \geq 0\} \\ s.t. \quad & \sum_{i=1}^{l} V_i \leq V_0 \\ & V_i = F_i^{-1} \left(\frac{b_i p_i - \mu}{b_i(h_i + p_i)} \right), \quad i = 1, 2, ..., l \end{aligned}$$

and Heuristic A replenishes item i up to V_i.

It is straightforward to extend Heuristic B to multi-item cases from two-item cases. Without loss of generality, assume that there are $l \geq 2$ and item 1 is replenished in period j. Then we can determine the adjusted capacities of items by solving the following equations:

$$\begin{aligned} & V_1 + \alpha E[a_j\ D_1] + \sum_{i=2}^{l} (V_i - (1-\alpha)E[e_{i,j}\ D_i]) = V_0 \\ & \alpha E[a_j\ D_1] - \sum_{i=2}^{l} (1-\alpha)E[e_{i,j}\ D_i] = 0 \end{aligned}$$

and we get

$$\begin{aligned} V_1' &= V_1 + \alpha E[a_j\ D_1] \\ V_i' &= V_i - (1-\alpha)E[e_{i,j}\ D_i], \quad i = 2, 3, ..., l \\ \alpha &= \frac{\sum_{i=2}^{l} e_{i,j}\ E[D_i]}{a_j\ E[D_1] + \sum_{i=2}^{l} e_{i,j}\ E[D_i]} \end{aligned}$$

The replenishment policy for Heuristic B remains the same as in the two-item case. No change is made for Heuristic C. We now summarize these three heuristics as follows:

Heuristic A

- Compute the separate capacity of each item by solving the following problem:

$$\begin{aligned} \textit{Find} \quad & \mu^* = \min\{\mu \geq 0\} \\ s.t. \quad & \sum_{i=1}^{l} V_i \leq V_0 \\ & V_i = F_i^{-1}\left(\frac{b_i p_i - \mu}{b_i(h_i + p_i)}\right), \quad i = 1, 2, ..., l \end{aligned}$$

- Determine the replenishment quantity

$$Q_{i,j} = V_i - I_{i,j} \ , \quad i = 1, 2, ...l$$

Heuristic B

- Compute the separate capacity of each item as for Heuristic A.
- Determine the number of periods between jth and $(j+1)$th replenishment in the system, a_j.
- Assuming that item 1 is replenished in period j, determine the number of preceding periods of item i since its last replenishment, $e_{i,j}$, $i = 2, 3, ..., l$.
- Adjust the capacities of individual items based on their demands as follows:

$$V_1' \;=\; V_1 + \alpha E[a_j \, D_1]$$

$$
\begin{aligned}
V_i' &= V_i - (1-\alpha)E[e_{i,j}\ D_i]\ , \quad i = 2, 3, \ldots l \\
\alpha &= \frac{\sum_{i=2}^{l} e_{i,j}\ E[D_i]}{a_j\ E[D_1] + \sum_{i=2}^{l} e_{i,j}\ E[D_i]}
\end{aligned}
$$

- Determine the replenishment quantity

$$
Q_{1,j} = \min\left[F_1^{-1}\left(\frac{p_1}{h_1+p_1}\right),\ V_1',\ V_0 - \sum_{i=2}^{l} I_{i,j}\right] - I_{1,j}
$$

Heuristic C

- Assuming that item 1 is replenished in period j, determine the available capacity, $V_0 - \sum_{i=2}^{l} I_{i,j}$, for item 1.
- Determine the replenishment quantity

$$
Q_{1,j} = \min\left[F_1^{-1}\left(\frac{p_1}{h_1+p_1}\right),\ V_0 - \sum_{i=2}^{l} I_{i,j}\right] - I_{1,j}
$$

3.4.4 Heuristics for Simultaneous Replenishment Case

The heuristics that we have developed so far are limited for the cases that only a single item is replenished in one period. In real warehouse systems, however, it is more likely that multiple items are replenished at the same time. The main issue in this case is how to allocate the available capacity to those items replenished simultaneously. For Heuristic A, the process to determine individual capacities remains the same as the single item replenishment cases since the heuristic depends on the number of items in the system, but not on the number of items replenished simultaneously. This is true because, in Heuristic A, we determine individual capacity of

each item under the assumption that all items are replenished at the same time and thus it does not affected by whether an item is replenished or not in a specific time instant.

To formulate the second heuristic, let $J = \{1, 2, ..., l\}$ be the index set of items in the system. We also define $R_j \subseteq J$ as the index set of items replenished in period j. Similar to the single replenishment case, we can determine the adjusted capacity of each item by solving the following equations:

$$\sum_{i \in R_j} (V_i + \alpha E[a_j \ D_i]) + \sum_{k \in J - R_j} (V_k - (1-\alpha)E[e_{k,j} \ D_k]) = V_0$$
$$\sum_{i \in R_j} \alpha E[a_j \ D_i] - \sum_{k \in J - R_j} (1-\alpha)E[e_{k,j} \ D_k] = 0$$

and we get

$$V_i' = V_i + \alpha E[a_j \ D_i], \quad i \in R_j$$
$$V_k' = V_k - (1-\alpha)E[e_{k,j} \ D_k], \quad k \in J - R_j$$
$$\alpha = \frac{\sum_{k \in J - R_j} e_{k,j} \ E[D_k]}{\sum_{i \in R_j} a_j \ E[D_i] + \sum_{k \in J - R_j} e_{k,j} \ E[D_k]}$$

We observed, in the single-item replenishment case, that Heuristic C had a replenishment policy which replenishes a single item up to the total available capacity in a period. We follow the same replenishment policy in this case. However, because of simultaneous replenishment, we need to allocate the total available capacity, $V_0 - \sum_{k \in J - R_j} I_{k,j}$, to individual items that are replenished in the same period. We first determine individual capacities of these items, as for the case of Heuristic A. Note that, in this case, we only compute the capacities of items replenished in a

period (say, period j) instead of determining the capacities of all items. The related expected cost in period j is

$$\sum_{i\in R_j} E\left[b_i\left(h_i(y_i-d_i)^+ + p_i(d_i-y_i)^+\right)\right]$$

where b_i is the number of replenishments of item i in one replenishment cycle. The optimal order-up-to level of item $i, i\in R_j$ under the warehouse constraint is then

$$\begin{aligned} y_i^* &= F_i^{-1}\left(\frac{b_ip_i-\mu^*}{b_i(h_i+p_i)}\right), \quad i\in R_j, \\ \mu^* &= arg \min_{\mu\geq 0}\left[\sum_{i\in R_j} F_i^{-1}\left(\frac{b_ip_i-\mu}{b_i(h_i+p_i)}\right) + \sum_{k\in J-R_j} I_{k,j} \leq V_0\right] \end{aligned}$$

and the capacity of each item can be obtained by solving the following equations:

$$\begin{aligned} &Find \quad && \mu^* = \min\{\mu\geq 0\} \\ &s.t. && \sum_{i\in R_j} V_i + \sum_{i'\in J-R_j} I_{i',j}+ \leq V_0 \\ & && V_i = F_i^{-1}\left(\frac{b_ip_i-\mu}{b_i(h_i+p_i)}\right), \quad i\in R_j \end{aligned}$$

Once the individual capacities are computed and assigned to each replenished item, we look into any unassigned storage. Since $\sum_{i\in R_j} V_i \leq V_0$, there might be a certain amount of unassigned space in the warehouse (we call thisleft-over capacity). For Heuristic C, we use up all available capacity for replenishment. Thus, when we replenish items, we use this left-over capacity as well as their own capacity. For example, when we replenish the first item (say, item i_1), the current left-over capacity is $LV_0 = V_0 - \sum_{i\in R_j} V_i - \sum_{i'\in J-R_j} I_{i',j}$ and the item is replenished up to the minimum

of $LV_0 + V_{i_1}$ and $F_{i_1}^{-1}\left(\frac{p_{i_1}}{h_{i_1}+p_{i_1}}\right)$. In case that $F_{i_1}^{-1}$ is smaller, the left-over capacity for the next item (say, item i_2) is $LV_1 = LV_0 + V_{i_1} - F_{i_1}^{-1}$. Otherwise, there is no left-over capacity available since the item is replenished up to $LV_0 + V_{i_1}$. In other words, the left-over capacity for item i_2 is $LV_2 = max\{0,\ LV_0 + V_{i_1} - F_{i_1}^{-1}\}$, and the item is replenished up to $min\{LV_2 + V_{i_2},\ F_{i_2}^{-1}\}$. We continue this process until all items are replenished. Note that when only one item is replenished in period j, then $LV_0 + V_{i_1} = V_0 - \sum_{i' \in J-R_j} I_{i',j}$ and the item is replenished up to $min\{V_0 - \sum_{i' \in J-R_j} I_{i',j},\ F_{i_2}^{-1}\}$, which is identical to the single-item replenishment case established in the previous section. We summarize these three heuristics in the following:

Heuristic A

- Compute the separate capacity of each item by solving the following equations:

$$\begin{aligned} Find \quad & \mu^* = \min\{\mu \geq 0\} \\ s.t. \quad & \sum_{i=1}^{l} V_i \leq V_0 \\ & V_i = F_i^{-1}\left(\frac{b_i p_i - \mu}{b_i(h_i + p_i)}\right), \quad i = 1, 2, ..., l \end{aligned} \tag{3.17}$$

- Determine the replenishment quantity

$$Q_{i,j} = V_i - I_{i,j}\ , \quad i \in R_j$$

Heuristic B

- Compute the separate capacity of each item, V_i, as for Heuristic A.
- Adjust the capacities of individual items based on their demands as follows:

· If item i is replenished in period j,

$$V_i' = V_i + \alpha E[a_j\ D_i], \quad i \in R_j$$

where

$$\alpha = \frac{\sum_{k \in J-R_j} e_{k,j}\ E[D_k]}{\sum_{i \in R_j} a_j\ E[D_i] + \sum_{k \in J-R_j} e_{k,j}\ E[D_k]}$$

· If item i is not replenished in period j,

$$V_i' = V_i - (1-\alpha) E[e_{i,j}\ D_i], \quad i \in J - R_j$$

- Determine the replenishment quantity

$$Q_{i,j} = \min \left[F_i^{-1}\left(\frac{p_i}{h_i + p_i}\right),\ V_i',\ V_0 - \sum_{k \in J-R_j} I_{k,j} \right] - I_{i,j}\ , \quad i \in R_j$$

Heuristic C

Let $R_j = \{i_1, i_2, ..., i_k\}$ where k is the number of items replenished in period j and i_k is the index of the kth item replenished in the period.

1. Determine the separate capacity of each replenished item by solving the following equations:

$$\begin{aligned} &\mathit{Find} \quad && \mu^* = \min\{\mu \geq 0\} \\ &s.t. && \sum_{i \in R_j} V_i + \sum_{i' \in J-R_j} I_{i',j}+ \leq V_0 \\ & && V_i = F_i^{-1}\left(\frac{b_i p_i - \mu}{b_i(h_i + p_i)}\right), \quad i \in R_j \end{aligned}$$

2. Compute any left-over capacity after the capacity allocation to each replenished item.

$$LV_0 = V_0 - \sum_{i \in R_j} V_i - \sum_{i' \in J - R_j} I_{i',j}$$

3. Replenish item i_1 up to

$$\min\{LV_0 + V_{i_1},\ F_{i_1}^{-1}\}$$

4. Compute any left-over capacity after replenishing item i_1.

$$LV_1 = \max\{0,\ LV_0 + V_{i_1} - F_{i_1}^{-1}\}$$

5. Repeat 3 & 4 for item $i_2, ..., i_k$

3.5 Numerical Results

This section presents numerical results illustrating the performance of the heuristics described in the previous section. Three different sets of replenishment schedules are considered. The first two sets are Schedules I and II which we mentioned before. The third set of schedules (Schedule III) is for the simultaneous replenishment case. For Schedule III, item 1 is replenished in periods 1, 3, 5, 7, ... , item 2 in periods 1, 4, 7, 11, ... and item 3 in periods 2, 5, 8, 11, ... We run a number of sample problems for each schedule by varying input parameters. Three different levels of unit cost parameters are chosen to be $h_i, p_i = 2, 5, i = 1, 2, 3$. Demand for each item follows an exponential distribution with parameter λ_i and we use three different levels of demand parameters for each item (λ_i=10, 30, 50). Since each item for Schedule I is replenished in every other period, the actual demand for one replenishment period

has an Erlang distribution with parameters $\beta = 2$ and $\lambda_i, i = 1, 2$ and the cumulative distribution function of the demand is then as follows:

$$\begin{aligned} F_i(x) &= \int_0^x \frac{x^{\beta-1}}{\Gamma(\beta)\lambda_i^\beta} e^{-\frac{s}{\lambda_i}} ds \\ &= 1 - \left(1 + \frac{x}{\lambda_i}\right) e^{-\frac{x}{\lambda_i}} \end{aligned}$$

For Schedule II, item 2 is replenished in every fourth period and thus its demand has an Erlang distribution with parameters $\beta = 4$ and λ_2 whose *cdf* is

$$F_2(x) = 1 - \left(1 + \frac{x}{\lambda_2} + \frac{x^2}{2\lambda_2^2} + \frac{x^3}{6\lambda_2^3}\right) e^{-\frac{x}{\lambda_2}}$$

and for Schedule III, items 2 & 3 are replenished in every third period so that the respective demand for both items have an Erlang distribution with parameters $\beta = 3$ and $\lambda_i, i = 2, 3$ whose *cdf* is

$$F_i(x) = 1 - \left(1 + \frac{x}{\lambda_i} + \frac{x^2}{2\lambda_i^2}\right) e^{-\frac{x}{\lambda_i}}$$

Unfortunately, in many cases, the inverse of F_i is intractable, so we employ a numerical approximation to compute F_i^{-1}.

The initial inventory level of each item is set to be $I_{1,1}, I_{2,1}, I_{3,1} = 10$, respectively. The choice of initial inventory levels should not have a major impact on the decision of replenishment policy in the long run. The system is simulated for $m = 50$ periods for each of the three heuristic policies using the same demand stream and the respective overall expected cost is recorded. These 50-period simulations are repeated 5,000 times.

It would be ideal to compare the expected cost of each heuristic policy to the expected cost of the optimal policy. Unfortunately, it is impractical to obtain dynamic programming solutions except for small sized problems with discrete demands and thus such a comparison would not be possible. Even though we do not try drawing any conclusions from the results obtained by running small sized problems, we might get insight from the results to provide some measure of the effectiveness of the heuristics developed in this study to the optimal policy. For this purpose, we run a two-item, two-period problem. Note that we have already shown, in the previous section, that the expected cost of this problem was convex in the replenishment quantity. Furthermore, referring to (3.11), the optimal solution can be obtained by solving the following first-order conditions:

$$
\begin{aligned}
&(h_1 + p_1)F_1(I_{1,1} + Q_1) - p_1 + \mu_1 \\
&\qquad\qquad + p_2 - (h_2 + p_2)\int_{d_{1,1}=0}^{a(Q_1)} F_{2,2}(b(Q_1)) f_{1,1} \mathrm{d}d_{1,1} = 0 \\
&\mu_1(I_{1,1} + I_{2,1} + Q_1 - V_0) = 0
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(x) &= 1 - \left(1 + \frac{x}{\lambda_1}\right) e^{-\frac{x}{\lambda_1}} \\
F_{2,2}(x) &= 1 - e^{-\frac{x}{\lambda_2}} \\
a(Q_1) &= F_{2,2}^{-1}\left(\frac{p_2}{h_2 + p_2}\right) - V_0 + I_{1,1} + Q_1 \\
b(Q_1) &= V_0 - I_{1,1} - Q_1 + d_{1,1}
\end{aligned}
$$

We solve these equations using numerical approximation since the solution is not tractable. Tables 3.1 $\sim$ 3.4 contain the average costs (over the planning horizon) of each heuristic policy as well as the optimal policy obtained from above equations. Each table reports the numerical results for different cost and demand parameters. The average costs presented in the tables are taken over 50 periods and over 5,000 replication. Figure 3.8 illustrates the graphical results of Tables 3.1 $\sim$ 3.4, respectively. As can be seen from these tables and figures, the cost ratio of each heuristic policy to the optimal policy varies with the warehouse capacity. Except for cases with very tight warehouse capacities, each heuristic policy retains an average cost close to the optimal policy. We also can observe that Heuristic A has a relatively higher average cost than Heuristics B and C in most cases. Furthermore, we can see that when the capacity increases and thus the capacity constraint becomes non-binding, these heuristics yield near-optimal expected costs. The latter result would be reasonable because, as noted in the previous section, each heuristic policy leads to the unconstrained optimal replenishment policy when the capacity constraint is not tight.

We now compare the performance of each proposed heuristic for more general cases. Numerical trials are conducted for the capacity of $V_0 = 100$ with relatively low demand parameters compared to the capacity. In the tables and figures below we illustrate the results of numerical tests for replenishment schedule I. Table 3.5 presents the averages of expected cost over all possible cost parameters for each setting of demand parameters, while Figure 3.9 illustrates the graphical results of

Table 3.1: Heuristic performance comparisons against the optimal policy for different capacities. $(\lambda_1, \lambda_2) = (30{,}50)$, $(h1, h2, p1, p2) = (2, 2, 2, 2)$.

Capacity	Average Cost						
	A	Ratio*	B	Ratio*	C	Ratio*	Optimal
20	288.1	1.17	259.9	1.06	256.0	1.04	246
40	254.1	1.12	231.5	1.02	230.9	1.02	227
60	230.5	1.04	225.9	1.02	225.9	1.02	223
80	221.8	1.00	222.7	1.00	222.7	1.00	222
100	221.5	1.00	221.5	1.00	221.5	1.00	221
120	221.5	1.00	221.5	1.00	221.5	1.00	221
Overall	239.6	1.05	230.5	1.02	229.8	1.01	227

* Cost/Ratio to Optimal

Table 3.2: Heuristic performance comparisons against the optimal policy for different capacities. $(\lambda_1, \lambda_2) = (50{,}50)$, $(h1, h2, p1, p2) = (2, 2, 2, 2)$.

Capacity	Average Cost						
	A	Ratio*	B	Ratio*	C	Ratio*	Optimal
40	330.2	1.19	288.5	1.04	288.5	1.04	278.4
60	296.7	1.14	268.3	1.03	269.3	1.04	259.5
80	273.4	1.06	261.2	1.02	261.1	1.02	257.2
100	259.7	1.01	257.9	1.01	257.9	1.01	256.3
120	256.3	1.00	256.3	1.00	256.3	1.00	256.3
Overall	283.3	1.08	266.4	1.02	266.6	1.02	261.6

* Cost/Ratio to Optimal

Table 3.3: Heuristic performance comparisons against the optimal policy for different capacities. $(\lambda_1, \lambda_2) = (30{,}10)$, $(h1, h2, p1, p2) = (2, 2, 5, 5)$.

Capacity	Average Cost						
	A	Ratio*	B	Ratio*	C	Ratio*	Optimal
40	225.3	1.23	197.6	1.08	197.6	1.08	183
60	181.3	1.08	168.8	1.01	169.4	1.01	168
80	176.5	1.06	166.8	1.00	166.8	1.00	166
100	166.3	1.00	166.3	1.00	166.3	1.00	166
120	166.3	1.00	166.3	1.00	166.3	1.00	166
Overall	183.1	1.07	173.2	1.02	173.3	1.02	170

* Cost/Ratio to Optimal

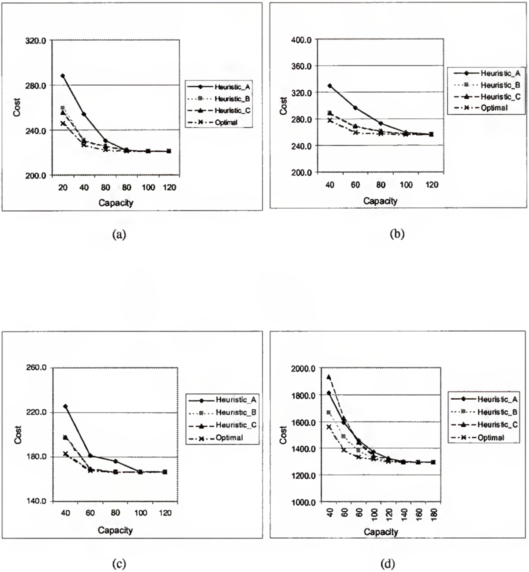


Figure 3.8: Heuristic performance comparisons against the optimal policy for different capacity. (a) (λ_1, λ_2) = (30,50), $(h1, h2, p1, p2)$ = (2, 2, 2, 2). (b) (λ_1, λ_2) = (50,50), $(h1, h2, p1, p2)$ = (2, 2, 2, 2). (c) (λ_1, λ_2) = (10,50), $(h1, h2, p1, p2)$ = (2, 2, 5, 5). (d) (λ_1, λ_2) = (50,10), $(h1, h2, p1, p2)$ = (2, 2, 20, 20).

Table 3.4: Heuristic performance comparisons against the optimal policy for different capacities. (λ_1, λ_2) = (10,50), $(h1, h2, p1, p2)$ = (2, 2, 20, 20).

Capacity	Average Cost						
	A	Ratio*	B	Ratio*	C	Ratio*	Optimal
40	1815.3	1.16	1664.5	1.07	1934.5	1.24	1558.5
60	1594.5	1.15	1487.4	1.07	1627.9	1.17	1389.6
80	1460.4	1.09	1381.8	1.04	1445.2	1.08	1333.8
100	1372.2	1.04	1331.3	1.01	1345.8	1.02	1321.8
120	1323.3	1.02	1311.4	1.01	1325.6	1.02	1301.8
140	1299.8	1.00	1297.1	1.00	1297.1	1.00	1296.5
160	1295.6	1.00	1295.6	1.00	1295.6	1.00	1295.6
180	1295.6	1.00	1295.6	1.00	1295.6	1.00	1295.6
Overall	1432.1	1.06	1383.1	1.02	1445.9	1.07	1349.2

* Cost/Ratio to Optimal

Table 3.5. Table 3.6 presents the average values of expected cost over all possible demand parameters for given specific cost parameters, and those results are graphically represented in Figure 3.10. As can be seen from Table 3.5 and Figure 3.9, for different demand rates, all three heuristics show almost identical overall performances. We also can notice, from Table 3.6 and Figure 3.10, that this result is retained for different cost parameters. This is true because, in most cases, the capacity constraint is not binding due to low demand rates and thus the problem becomes unconstrained. Without the capacity constraint, all three heuristics have the identical replenishment policy which replenishes products up to the optimal inventory level in the unconstrained system, as mentioned in the previous section. This result is consistent with replenishment schedules II and III (we omit the results for these schedules). Since it is hardly possible to identify which heuristic shows the best performance, we next consider the case with the capacity being tight to see if any heuristic outperforms the others.

Table 3.5: Heuristic performance for different demand parameters.

Demand		Average Cost				
λ_1	λ_2	A	Ratio*	B	Ratio*	C
10	10	1398.3	1.00	1398.3	1.00	1398.3
10	20	2146.4	1.00	2146.4	1.00	2146.4
10	30	2892.6	1.00	2892.6	1.00	2892.6
20	10	2052.3	1.00	2052.3	1.00	2052.3
20	20	2800.5	1.00	2800.5	1.00	2800.5
20	30	3581.3	1.00	3551.5	0.99	3586.6
30	10	2705.5	1.00	2705.5	1.00	2705.5
30	20	3487.3	1.00	3457.4	0.99	3479.9
Overall		2633.0	1.00	2625.6	1.00	2632.8

* Cost/Ratio to Heuristic C

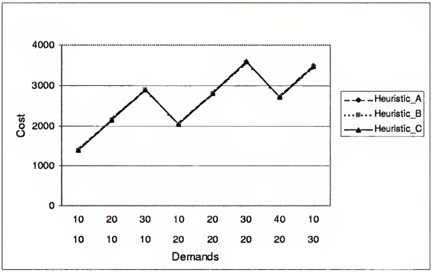


Figure 3.9: Heuristic performance for different demand parameters.

Table 3.6: Heuristic performance for different cost parameters.

Unit Costs				Average Cost				
h1	h2	p1	p2	A	Ratio*	B	Ratio*	C
2	2	2	2	2092.6	1.00	2092.6	1.00	2092.4
2	2	5	5	3799.5	1.00	3681.4	0.97	3787.5
2	5	2	5	3573.3	1.00	3573.2	1.00	3573.2
5	2	5	2	3750.9	1.00	3750.7	1.00	3750.1
5	5	2	2	2693.5	1.00	2693.5	1.00	2693.5

* Cost/Ratio to Heuristic C

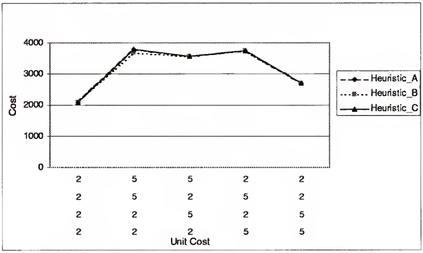


Figure 3.10: Heuristic performance for different cost parameters.

In new simulation runs, we vary the capacity while cost and demand parameters are fixed. The choice of various capacities ranges from 20 to 400. This capacity range seems to be reasonable for our purpose because when the capacity is 20, the constraint is likely tight in most cases with the aforementioned demand rates, and when it is 400, the constraint would most likely be non-binding. Henceforth, we can identify the performance of each heuristic whether the capacity constraint is tight or non-binding.

We generate a number of problems from the set of cost and demand parameters. Tables 3.7 $\sim$ 3.11 present average costs for different capacities for replenishment schedule I. Tables 3.12 $\sim$ 3.16 report the results for Schedule II. The results for Schedule III are shown in Tables 3.17 $\sim$ 3.21. In this study we present computational results for five different problems for each replenishment schedule. The results for other problems retain a similar pattern to those in the above tables, and thus we omit them here.

In these tables, we compare the performance of each heuristic to the unconstrained optimal replenishment policy with respect to the expected cost over the planning horizon. A/U, B/U and C/U represent the cost-ratios of heuristics A, B and C to the unconstrained optimal policy, respectively. We also compare the performance of heuristics B and C to Heuristic A, which we consider as a base since the heuristic is the most conservative one.

Note that when there is a sufficient amount of slack in the resource constraint, all heuristics seem to have identical performance, and thus the cost-ratio of each heuristic to the unconstrained optimal policy approaches to one. This is because,

when the system becomes unconstrained, all of them employ the same replenishment policy that replenished products up to the unconstrained optimal replenishment level.

As can be seen from Tables 3.7 through 3.9 for Schedule I, the impact of tightening the capacity constraint is significant. When resources are scarce, Heuristics B and C have better performance than Heuristic A (Heuristic A has 4-11 % higher average cost than the other two). This behavior is not surprising, especially since Heuristic A holds the smallest individual capacity for each item and it is more likely to retain unused capacity for some items, while some other items need more storage to satisfy their customer demands, causing unnecessary penalty costs. Heuristic C, however, does not carry this capacity-inefficiency incurred in Heuristic A because it uses the total available capacity whenever the warehouse replenishes products. Heuristic B would confront the same pitfall as Heuristic A because it also adopts separate capacities for replenishment, but this flaw is lessened by the capacity-adjustment characteristic of the heuristic.

On the other hand, Tables 3.10 and 3.11 contain results for a special case with high penalty costs ($p_i = 20$). Obviously, Heuristic C leads to poorer performance when the capacity is getting tight. The cost ratio of Heuristic C to Heuristic A ranges from 0.98 to 1.18, depending on the capacity, while the ratio of Heuristic B to Heuristic A is as low as 0.86. This is most likely due to the fact that, when the penalty cost of each product ($p_i = 20$) is relatively high with respect to the holding cost ($h_i = 2$), this heuristic is inclined to hold excessive inventories for products being replenished in a period in order to avoid the expensive penalty cost, and it leaves little slack in the resource constraint in the following replenishment period,

causing high penalty costs for the products to be replenished in the next period. Heuristics A and B, however, are limited in using any available capacity because of using separate capacity for each product, so that both of the heuristics still keep some space for the next replenishment and do not carry the same degree of severity as Heuristic C does.

3.6 Conclusions

In this chapter, we generalize the multi-item, periodic-review inventory model described in the previous chapter, by relaxing the assumption of identical replenishment schedules for items. We first completely characterized the optimal replenishment policy for the two-item, two-period case. For more general cases, because of the inability to show convexity of the objective function, we construct three heuristics to determine replenishment quantities for the case of unequal replenishment intervals.

Numerical testing of these heuristics suggests that they yield near optimal solutions for a small set of problems. The performance of these heuristics has been evaluated for a number of problems with various cost and demand parameters. When there is enough slack in the resource constraint, all heuristics demonstrate identical results.

The first heuristic, denoted Heuristic A, which uses a separate capacity for each product for replenishment, is simple enough for practical use, but is not able to detect a scarce resource well, compared to the other two heuristics, in most problems. This behavior is plausible because the heuristic is more likely to retain unused capacities for some products while some others suffer from the lack of the resource.

Heuristic C adopts a replenishment policy which replenishes products up to the total available capacity of the system at the time instant of replenishment. This heuristic outperforms Heuristic A in most cases because it does not carry the capacity-inefficiency confronted by the Heuristic A , but leads to poorer performance for the problems with high penalty costs. This is because, in this situation, Heuristic C is too aggressive to replenish products in one period so that the warehouse suffers from the lack of resource in the following replenishment period, causing high penalty costs to the warehouse.

On the other hand, the second heuristic, denoted Heuristic B, adjusts separate capacities in each replenishment period, based on the availability of any unused resource. This characteristic discriminates Heuristic B from Heuristic A and diminishes the capacity-inefficiency of Heuristic A. Furthermore, the capacity-separation scheme of the heuristic leads to better performance than Heuristic C.

Table 3.7: Heuristic performance for different capacities with Schedule I. $(\lambda_1, \lambda_2) =$ (30,30), $(h1, h2, p1, p2) =$ (2, 2, 2, 2).

	Cost/Ratio				
Capacity	A/U	B/U	C/U	B/A	C/A
20	1.80	1.60	1.67	0.89	0.93
40	1.46	1.36	1.41	0.93	0.97
60	1.21	1.13	1.16	0.94	0.96
80	1.05	1.05	1.05	1.00	1.00
100	1.00	1.00	1.00	1.00	1.00
120	1.00	1.00	1.00	1.00	1.00
140	1.00	1.00	1.00	1.00	1.00
Overall	1.30	1.23	1.26	0.95	0.97

Table 3.8: Heuristic performance for different capacities with Schedule I. $(\lambda_1, \lambda_2) =$ (30,30), $(h1, h2, p1, p2) =$ (5,5,2,2).

	Cost/Ratio				
Capacity	A/U	B/U	C/U	B/A	C/A
20	1.39	1.25	1.26	0.90	0.90
40	1.15	1.08	1.08	0.94	0.94
60	1.01	1.00	1.00	0.99	0.99
80	1.00	1.00	1.00	1.00	1.00
100	1.00	1.00	1.00	1.00	1.00
120	1.00	1.00	1.00	1.00	1.00
Overall	1.14	1.08	1.08	0.96	0.96

Table 3.9: Heuristic performance for different capacities with Schedule I. $(\lambda_1, \lambda_2) =$ (10,50), $(h1, h2, p1, p2) =$ (2, 2, 2, 2).

	Cost/Ratio				
Capacity	A/U	B/U	C/U	B/A	C/A
20	1.72	1.64	1.65	0.95	0.96
40	1.42	1.35	1.36	0.95	0.96
60	1.19	1.16	1.17	0.97	0.98
80	1.05	1.03	1.05	0.98	1.00
100	1.00	1.00	1.00	1.00	1.00
120	1.00	1.00	1.00	1.00	1.00
140	1.00	1.00	1.00	1.00	1.00
Overall	1.28	1.24	1.24	0.97	0.98

Table 3.10: Heuristic performance for different capacities with Schedule I. $(\lambda_1, \lambda_2) =$ (10,50), $(h1, h2, p1, p2)$ = (2, 2, 20, 20).

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
20	5.18	4.90	5.69	0.94	1.10
40	4.21	3.94	4.71	0.94	1.12
60	3.37	3.16	3.75	0.94	1.11
80	2.67	2.51	2.95	0.94	1.10
100	2.12	2.02	2.30	0.95	1.08
120	1.72	1.65	1.82	0.96	1.06
140	1.42	1.38	1.49	0.97	1.04
160	1.22	1.21	1.26	0.99	1.03
180	1.09	1.09	1.12	0.99	1.02
200	1.02	1.02	1.04	1.00	1.02
220	1.00	1.00	1.00	1.00	1.00
240	1.00	1.00	1.00	1.00	1.00
260	1.00	1.00	1.00	1.00	1.00
Overall	2.17	2.07	2.34	0.97	1.06

Table 3.11: Heuristic performance for different capacities with Schedule I. $(\lambda_1, \lambda_2) =$ (30,30), $(h1, h2, p1, p2)$ = (2, 2, 20, 20).

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
Capacity	A	Ratio*	B	Ratio*	C
20	5.48	4.85	5.39	0.89	0.98
40	4.42	3.79	4.55	0.86	1.03
60	3.50	3.01	3.81	0.86	1.09
80	2.75	2.40	3.14	0.87	1.14
100	2.17	1.93	2.56	0.89	1.18
120	1.74	1.58	2.04	0.91	1.17
140	1.44	1.33	1.58	0.92	1.10
160	1.23	1.16	1.30	0.95	1.06
180	1.09	1.07	1.13	0.98	1.04
200	1.02	1.02	1.04	1.00	1.02
220	1.00	1.00	1.00	1.00	1.00
240	1.00	1.00	1.00	1.00	1.00
260	1.00	1.00	1.00	1.00	1.00
Overall	2.24	2.01	2.38	0.93	1.07

Table 3.12: Heuristic performance for different capacities with Schedule II. (λ_1, λ_2) = (30,30), $(h1, h2, p1, p2)$ = (2, 2, 2, 2).

Capacity	Cost/Ratio A/U	B/U	C/U	B/A	C/A
40	1.97	1.78	1.72	0.90	0.87
60	1.65	1.57	1.57	0.95	0.95
80	1.56	1.46	1.46	0.94	0.94
100	1.52	1.46	1.46	0.96	0.96
120	1.11	1.07	1.07	0.97	0.97
140	1.07	1.04	1.04	0.97	0.97
160	1.00	1.00	1.00	1.00	1.00
180	1.00	1.00	1.00	1.00	1.00
Overall	1.41	1.34	1.33	0.96	0.95

Table 3.13: Heuristic performance for different capacities with Schedule II. (λ_1, λ_2) = (10,50), $(h1, h2, p1, p2)$ = (5, 5, 2, 2).

Capacity	Cost/Ratio A/U	B/U	C/U	B/A	C/A
40	1.77	1.71	1.68	0.96	0.95
60	1.61	1.53	1.51	0.96	0.94
80	1.43	1.36	1.36	0.95	0.95
100	1.34	1.31	1.31	0.98	0.98
120	1.33	1.31	1.31	0.99	0.99
140	1.02	1.01	1.01	0.99	0.99
160	1.00	1.00	1.00	1.00	1.00
Overall	1.36	1.32	1.31	0.98	0.97

Table 3.14: Heuristic performance for different capacities with Schedule II. (λ_1, λ_2) = (50,30), $(h1, h2, p1, p2)$ = (2, 2, 2, 2).

Capacity	Cost/Ratio A/U	B/U	C/U	B/A	C/A
60	1.77	1.59	1.57	0.90	0.88
80	1.54	1.44	1.45	0.94	0.94
100	1.45	1.36	1.36	0.94	0.94
120	1.24	1.19	1.19	0.96	0.97
140	1.11	1.09	1.09	0.98	0.98
160	1.04	1.04	1.04	1.00	1.00
180	1.00	1.00	1.00	1.00	1.00
Overall	1.31	1.24	1.24	0.96	0.96

Table 3.15: Heuristic performance for different capacities with Schedule II. $(\lambda_1, \lambda_2) = (30,10)$, $(h1, h2, p1, p2) = (2, 2, 20, 20)$.

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
60	3.47	3.13	4.65	0.90	1.34
80	2.97	2.73	3.46	0.92	1.16
100	2.73	2.57	2.89	0.94	1.06
120	1.33	1.30	1.49	0.97	1.12
140	1.11	1.13	1.24	1.02	1.12
160	1.02	1.06	1.07	1.04	1.05
180	1.01	1.01	1.01	1.01	1.01
200	1.00	1.00	1.00	1.00	1.00
220	1.00	1.00	1.00	1.00	1.00
Overall	1.83	1.74	2.10	0.98	1.11

Table 3.16: Heuristic performance for different capacities with Schedule II. $(\lambda_1, \lambda_2) = (50,30)$, $(h1, h2, p1, p2) = (2, 2, 20, 20)$.

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
60	3.27	2.91	3.65	0.89	1.11
80	2.61	2.33	2.88	0.89	1.10
100	2.11	1.91	2.24	0.91	1.06
120	1.72	1.58	1.79	0.92	1.05
140	1.47	1.40	1.49	0.95	1.02
160	1.41	1.37	1.36	0.97	0.96
180	1.10	1.10	1.17	1.00	1.06
200	1.03	1.04	1.08	1.02	1.05
220	1.01	1.03	1.02	1.02	1.01
240	1.00	1.01	1.01	1.01	1.01
260	1.00	1.00	1.00	1.00	1.00
280	1.00	1.00	1.00	1.00	1.00
Overall	1.61	1.52	1.70	0.96	1.04

Table 3.17: Heuristic performance for different capacities with Schedule III. $(\lambda_1, \lambda_2, \lambda_3) = (30,10,30)$, $(h1, h2, h3, p1, p2, p3) = (2, 2, 2, 2, 2, 2)$.

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
50	1.55	1.34	1.41	0.87	0.91
70	1.41	1.29	1.28	0.91	0.91
90	1.19	1.07	1.14	0.90	0.96
110	1.08	1.02	1.05	0.94	0.97
130	1.02	1.00	1.01	0.98	0.99
150	1.00	1.00	1.00	1.00	1.00
170	1.00	1.00	1.00	1.00	1.00
190	1.00	1.00	1.00	1.00	1.00
Overall	1.18	1.10	1.13	0.94	0.96

Table 3.18: Heuristic performance for different capacities with Schedule III. $(\lambda_1, \lambda_2, \lambda_3) = (10,10,50)$, $(h1, h2, h3, p1, p2, p3) = (2, 2, 2, 2, 2, 2)$.

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
50	1.61	1.42	1.49	0.88	0.93
70	1.41	1.26	1.31	0.89	0.93
90	1.26	1.14	1.17	0.91	0.93
110	1.15	1.12	1.12	0.97	0.97
130	1.13	1.11	1.11	0.99	0.99
150	1.04	1.02	1.02	0.99	0.99
170	1.00	1.00	1.00	1.00	1.00
190	1.00	1.00	1.00	1.00	1.00
Overall	1.23	1.15	1.17	0.95	0.96

Table 3.19: Heuristic performance for different capacities with Schedule III. $(\lambda_1, \lambda_2, \lambda_3) = (10,10,30)$, $(h1, h2, h3, p1, p2, p3) = (2, 2, 2, 5, 5, 5)$.

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
30	2.43	2.03	2.19	0.84	0.90
50	2.01	1.74	1.80	0.87	0.90
70	1.80	1.64	1.63	0.91	0.91
90	1.69	1.59	1.55	0.94	0.92
110	1.21	1.09	1.23	0.90	1.02
130	1.08	1.02	1.12	0.95	1.04
150	1.02	1.01	1.02	0.99	1.00
170	1.00	1.00	1.00	1.00	1.00
190	1.00	1.00	1.00	1.00	1.00
Overall	1.53	1.39	1.44	0.92	0.96

Table 3.20: Heuristic performance for different capacities with Schedule III. $(\lambda_1, \lambda_2, \lambda_3)$ = (10,10,10), $(h1, h2, h3, p1, p2, p3)$ = (2, 2, 2, 20, 20, 20).

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
30	4.26	3.40	5.38	0.80	1.26
50	3.02	2.41	4.01	0.80	1.33
70	2.14	1.74	2.99	0.81	1.40
90	1.58	1.33	2.67	0.84	1.69
110	1.21	1.10	1.59	0.91	1.31
130	1.03	1.01	1.03	0.98	1.00
150	1.00	1.00	1.00	1.00	1.00
Overall	2.04	1.71	2.67	0.88	1.28

Table 3.21: Heuristic performance for different capacities with Schedule III. $(\lambda_1, \lambda_2, \lambda_3)$ = (30,30,30), $(h1, h2, h3, p1, p2, p3)$ = (2, 2, 2, 20, 20, 20).

Capacity	Cost/Ratio				
	A/U	B/U	C/U	B/A	C/A
50	5.28	4.29	5.23	0.81	0.99
70	4.75	3.82	4.71	0.80	0.99
90	4.26	3.40	4.22	0.80	0.99
110	3.83	3.06	3.72	0.80	0.97
130	3.37	2.79	3.35	0.83	0.99
150	3.04	2.58	3.00	0.85	0.99
170	2.70	2.16	3.43	0.80	1.27
190	2.39	1.93	3.21	0.81	1.34
210	2.12	1.73	3.03	0.82	1.43
230	1.91	1.58	2.89	0.82	1.51
250	1.71	1.44	2.78	0.84	1.62
270	1.56	1.33	2.68	0.85	1.72
290	1.41	1.23	2.39	0.87	1.69
310	1.30	1.16	1.96	0.89	1.50
330	1.22	1.10	1.58	0.91	1.30
350	1.14	1.06	1.30	0.93	1.14
370	1.09	1.03	1.11	0.95	1.02
390	1.04	1.01	1.03	0.97	0.99
410	1.01	1.00	1.00	0.99	0.99
430	1.00	1.00	1.00	1.00	1.00
450	1.00	1.00	1.00	1.00	1.00
Overall	2.31	1.94	2.68	0.87	1.22

CHAPTER 4
CUSTOMER ORDER LEAD-TIME MODELS

4.1 Introduction

Uncertainties in manufacturing environments play important roles in determining planned customer order leadtimes. These uncertainties are mostly due to complicated production processes, random yields, and high quality requirements. Incorporation of these factors into the selection of planned leadtimes represents an important step toward increasing the robustness of manufacturing planning. The leadtime uncertainty itself affects many aspects of costs and control. It is particularly problematic in queueing systems because tardiness in earlier orders may delay subsequent orders.

Most of the due-date-setting models assume that the due dates for individual orders are set entirely exogenously. However, in certain practical contexts, each order arrives with a due date, indicating some future time when the customer wishes to receive the goods ordered and, in most cases, due date setting is negotiable and is the responsibility of the marketing personnel of the firm. For the marketing group, who have knowledge of customer's wishes, it is important to know their manufacturing personnel's perspective on the order before negotiating the due date with their customer. That is, they would like to know the optimal customer order leadtime to quote for the order. By *customer order leadtimes*, we mean the time from a customer's order until the due date.

This study is stimulated by Yano's work on PLTs in serial production systems (1987a). Yano developed a solution approach to solve the two-stage problem and applied it to N-stage serial systems. The limitation of their approach was that they did not address the effects of queueing or capacity explicitly in the model. For instance, Yano assumes that an order that has completed processing in one stage may be held until its due date (i.e. no early delivery to customers is allowed). However, it is also assumed that once that order arrives at the next stage, it is processed immediately without being delayed by preceded orders. This assumption greatly simplifies their solution approach for determining PLTs in serial production systems. The assumption used by Yano is not pragmatic in many situations. Literature such as Duenyas (1995) and Duenyas and Hopp (1995) considered queueing effects in determining PLTs. In this study, we consider the aforementioned delay as well as the possible holding cost incurred due to the prohibition of early delivery to customers in single-stage systems.

This study addresses single-stage make-to-order (MTO) production systems. Our main objective is to determine the optimal planned customer leadtimes while minimizing the sum of expected inventory-holding costs and expected penalty costs for exceeding the quoted due date, so as to quote planned leadtimes to customers at the time of order arrival. Since the area of modeling leadtime setting where demand is sensitive to quoted leadtimes is relatively unexposed, it would not be a critical shortcoming to assume that the leadtimes quoted to customers are independent of each other. Within this scope, we focus on modeling the appropriate expected costs under a variety modeling assumptions and characterizing the optimal policies.

Our approach to solving the problem is to derive the distribution of actual completion time of the process for an individual order and to compare this to the corresponding quoted due dates to obtain the expected total costs. We then minimize costs with respect to the decision variable(s), which, in this case, are PLT(s). We show that, in special cases, the single stage model is equivalent to a simple newsboy problem as Weeks (1981) pointed out.

4.2 Model Description and Formulation

Consider the problem of determining optimal planned leadtimes in single stage make-to-order production systems where individual orders wait in a $G/G/1$ queue until preceded orders complete their processes. The production system consists of one reliable machine and produces a single item (see Figure 4.1). Once the process for an order has been completed, the finished goods are held in inventory until the due date. If a job has already been tardy, the products are dispatched immediately (see Figure 4.2). Therefore, there are two possible delays, one by holding products until the due date and the other by waiting until the previous order is completed. The order size is a random variable.

Let k be the index of newly arrived order. Denote

x_k : planned customer leadtime of the kth order,
R : random variable representing yield rates with a density $g_1(r)$, $R > 0$,
Q_k : the size of order k, where $Q_k \sim iid$ a random variable with a density $g_2(q), Q_k = 1, 2, \cdots$,
t_k : arrival time of order k, a constant,
τ: processing time of unit product with a density $g_3(\tau)$,
s_k : actual process completion time of the kth order,
h : holding cost per unit product and per unit time,
p : penalty cost for late delivery per unit product and per unit time.

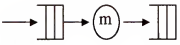


Figure 4.1: One machine, single part-type system.

Then the due date of order k, D_k, becomes $D_k = x_k + t_k$ and the total processing time of the kth order, τ_k, is $\tau_k = \frac{Q_k}{R}\tau$. The term $\frac{Q_k}{R}$ represents the actual amount of finished goods to produce to meet customer demands. The arrival time of that order, t_k, is known and constant. The actual process starting time of order k is $\max\{t_k, s_{k-1}\}$, where s_{k-1} is the job completion time of previous $(k-1)$th order (see Figure 4.2). This is true because if there is no order in the system, the newly arrived one will be served immediately. Otherwise, it must wait until the preceding order, order $(k-1)$, is completed. Then, the actual process completion time of order k, s_k, can be represented as follows:

$$s_k = \max\{t_k, s_{k-1}\} + \tau_k \tag{4.1}$$

Since an individual customer will not take a delivery before the due date, a product holding cost is incurred for early completion. The cost to customers of early deliveries are widely appreciated, partly due to the JIT movement. On the other hand, a tardy job will be released immediately after completion with a penalty cost. Under the above assumptions, we minimize the expected cost function with respect to the customer planned leadtime of order k, x_k,

$$\min_{x_k} \;\; \psi(x_k) = E[hQ_k(D_k - s_k)^+ + pQ_k(s_k - D_k)^+] \tag{4.2}$$

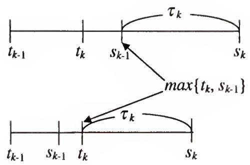


Figure 4.2: Configuration of process.

where

$$a^+ = \max\{a, 0\}$$

$$\begin{aligned} E[hQ_k(D_k - s_k)^+] &= E[hQ_k(t_k + x_k - s_k)^+] \\ &= E[hQ_k(t_k + x_k - \max\{t_k, s_{k-1}\} - \tau_k)^+] \\ E[hQ_k(s_k - D_k)^+] &= E[pQ_k(s_k - t_k - x_k)^+] \\ &= E[pQ_k(\max\{t_k, s_{k-1}\} + \tau_k - t_k - x_k)^+] \end{aligned}$$

The expectation is taken with respect to the random variables R, Q_k, τ and s_k. The first term in the right hand side in (4.2) denotes the expected holding cost due to early production and the second one represents the expected cost for tardiness. To solve this nonlinear optimization problem, we first derive the distribution of s_k.

4.3 Distribution of s_k

Since s_k is a function of $\max\{t_k, s_{k-1}\}$, we consider two possible cases separately. The first case is such that the system is empty when order k arrives, or $t_k > s_{k-1}$. In this case, the process completion time of the $(k-1)$th order is a known constant

and s_k is just sum of a random variable, τ_k, and a constant t_k.

Theorem 4.1 *s_k has the same distribution as τ_k with a location parameter t_k, where $t_k > \tau_k$.*

Proof : From (4.1), we get

$$s_k = \max\{t_k, s_{k-1}\} + \tau_k = t_k + \tau_k \quad \square$$

Since t_k is a known constant and the distribution of τ_k is given, we can easily derive the distribution of s_k.

Now we can simplify the expected cost function (4.2) as follows:

$$\begin{aligned}\psi(x_k) &= E\left[hQ_k(t_k + x_k - \max\{t_k, s_{k-1}\} - \tau_k)^+\right] \\ &\quad + E\left[pQ_k(\max\{t_k, s_{k-1}\} + \tau_k - t_k - x_k)^+\right] \\ &= E\left[hQ_k(x_k - \tau_k)^+\right] + E\left[pQ_k(\tau_k - x_k)^+\right] \qquad (4.3)\end{aligned}$$

The second case is such that t_k is less than or equal to s_{k-1}. In other words, there is at least one customer order in the system when order k arrives. Without lose of generality, let's assume that l orders, including the newly arrive order k, are in the system at time t_k. Then, s_k is a function of the total processing time of the last $k-1$ orders, the remaining processing time of the order in process at time t_k and a known constant, s_{k-l}, which is the process completion time of the latest order to finish at t_k.

Theorem 4.2 *The distribution of s_k is a convolution of a constant and random variables whose distributions are known*

Proof: Since there are l orders in the system, we get $t_k < s_k$, $t_k < s_{k-1}$, ..., $t_k < s_{k-l+1}$ and $t_k > s_{k-l}$. The first l inequalities result from the fact that the last l orders in the system have not been served yet at time t_k. Thus, the job completion time of those orders must greater than t_k. The last inequality is obvious because the $(k-l)$th order has already been processed at time t_k (the $(k-l+1)$th order is currently on processing) and, thus, s_{k-l} is a known constant at time t_k. We also know that $t_{k-l+1} < t_{k-l+2} < \cdots < t_{k-1} < t_k$. Combining these two sets of inequalities, we get

$$t_{k-l+1} < s_{k-l}, \qquad t_{k-l+2} < s_{k-l+1}, \quad \ldots \quad t_k < s_{k-1} \tag{4.4}$$

Applying (4.4) to (4.1), we have

$$\begin{aligned}
s_k &= \max\{t_k, s_{k-1}\} + \tau_k = s_{k-1} + \tau_k \\
s_{k-1} &= \max\{t_{k-1}, s_{k-2}\} + \tau_{k-1} = s_{k-2} + \tau_{k-1} \\
&\vdots \\
s_{k-l+1} &= \max\{t_{k-l+1}, s_{k-l}\} + \tau'_{k-l+1} = s_{k-l} + \tau'_{k-l+1} \\
\Rightarrow s_k &= s_{k-l} + \sum_{i=0}^{l-2} \tau_{k-i} + \tau'_{k-l+1}
\end{aligned} \tag{4.5}$$

where $\tau'_{k-l+1} = U + \tau_{k-l+1} - t_k$ is the remaining process time of order $(k-l+1)$ at time t_k, U is the actual process starting time of order $(k-l+1)$. Note that

the job completion time of order k, s_k, is a function of the job completion time of the $(k-l)$th order, s_{k-l}, and the projected total processing time of orders waiting in queue at time t_k. Now, we can derive the distribution of s_k from (4.5) since all random variables included in the right hand side in (4.5) have known distributions and s_{k-l} is a known constant. □

Denoting the density of s_k be $g_4(s_k)$ we can rewrite the expected cost function as follows:

$$\begin{aligned} \psi(x_k) &= E\left[hQ_k(t_k + x_k - s_k)^+\right] + E\left[pQ_k(s_k - t_k - x_k)^+\right] \\ &= h\int_{q_k} q_k g_2(q_k) \int_{s_k} (t_k + x_k - s_k)^+ g_4(s_k) ds_k dq_k \\ &\quad + p\int_{q_k} q_k g_2(q_k) \int_{s_k} (s_k - t_k - x_k)^+ g_4(s_k) ds_k dq_k \end{aligned} \tag{4.6}$$

4.4 Solution Approach

Once we derive distribution of s_k, the nonlinear optimization problem (4.2) can be solved separately for each of aforementioned cases. We establish a closed-form solution for each case and investigate the effects of changing the parameters in our model on the optimal planned customer leadtime.

4.4.1 Case I: $t_k > s_{k-1}$

In case that there are no orders waiting or on processing in the system when a new order arrives at time t_k, we get, from (4.3),

$$\psi(x_k) = E\left[hQ_k(x_k - \tau_k)^+\right] + E\left[pQ_k(\tau_k - x_k)^+\right]$$

$$
\begin{aligned}
&= E\left[hQ_k\left(x_k - \tau\frac{Q_k}{R}\right)^+\right] + E\left[pQ_k\left(\tau\frac{Q_k}{R} - x_k\right)^+\right] \\
&= h\int_r g_1(r)\int_{q_k} q_k g_2(q_k)\int_{\tau=0}^{\infty}\left(x_k - \frac{\tau q_k}{r}\right)^+ g_3(\tau)d\tau dq_k dr \\
&\quad +p\int_r g_1(r)\int_{q_k} q_k g_2(q_k)\int_{\tau=0}^{\infty}\left(\frac{\tau q_k}{r} - x_k\right)^+ g_3(\tau)d\tau dq_k dr \\
&= h\int_r g_1(r)\int_{q_k} q_k g_2(q_k)\int_{\tau=0}^{rx_k/q_k}\left(x_k - \frac{\tau q_k}{r}\right) g_3(\tau)d\tau dq_k dr \\
&\quad +p\int_r g_1(r)\int_{q_k} q_k g_2(q_k)\int_{\tau=rx_k/q_k}^{\infty}\left(\frac{\tau q_k}{r} - x_k\right) g_3(\tau)d\tau dq_k dr \qquad (4.7)
\end{aligned}
$$

and we want to minimize (4.7) with respect to the planned customer leadtime, x_k. Assuming $\psi(x_k)$ is twice differentiable, we can then prove the following:

Proposition 4.1 *$\psi(x_k)$ is convex in $x_k > 0$*

Proof : Differentiating (4.7) with respect to x_k, we obtain the following optimality conditions:

$$
\begin{aligned}
\frac{\partial\psi(x_k)}{\partial x_k} &= h\int_r g_1(r)\int_{q_k} q_k g_2(q_k)\frac{\partial}{\partial x_k}\left[\int_{\tau=0}^{rx_k/q_k}\left(x_k - \frac{\tau q_k}{r}\right) g_3(\tau)d\tau\right] dq_k dr \\
&\quad +p\int_r g_1(r)\int_{q_k} q_k g_2(q_k)\frac{\partial}{\partial x_k}\left[\int_{\tau=rx_k/q_k}^{\infty}\left(\frac{\tau q_k}{r} - x_k\right) g_3(\tau)d\tau\right] dq_k dr
\end{aligned}
$$

$$
\begin{aligned}
&= h\int_r g_1(r)\int_{q_k} q_k g_2(q_k)\int_{\tau=0}^{rx_k/q_k} g_3(\tau)d\tau dq_k dr \\
&\quad +p\int_r g_1(r)\int_{q_k} q_k g_2(q_k)\int_{\tau=rx_k/q_k}^{\infty} -g_3(\tau)d\tau dq_k dr \\
&= \int_r g_1(r)\int_{q_k} q_k g_2(q_k)\left[h\int_{\tau=0}^{rx_k/q_k} g_3(\tau)d\tau - p\int_{\tau=rx_k/q_k}^{\infty} g_3(\tau)d\tau\right] dq_k dr
\end{aligned}
$$

$$\begin{aligned}
&= \int_r g_1(r) \int_{q_k} q_k g_2(q_k) \left[(h+p) \int_{\tau=0}^{rx_k/q_k} g_3(\tau)d\tau - p \int_{\tau=0}^{\infty} g_3(\tau)d\tau \right] dq_k dr \\
&= \int_r g_1(r) \int_{q_k} q_k g_2(q_k) \left[(h+p) \int_{\tau=0}^{rx_k/q_k} g_3(\tau)d\tau - p \right] dq_k dr \\
&= (h+p) \int_r g_1(r) \int_{q_k} q_k g_2(q_k) \int_{\tau=0}^{rx_k/q_k} g_3(\tau) d\tau dq_k dr - pE[Q_k] \\
&= (h+p) \int_r g_1(r) \int_{q_k} q_k G_3(rx_k/q_k) g_2(q_k) dq_k dr - pE[Q_k] = 0 \qquad (4.8)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \psi(x_k)}{\partial {x_k}^2} &= (h+p) \int_r g_1(r) \int_{q_k} q_k \left[\frac{\partial}{\partial x_k} G_3(rx_k/q_k) \right] g_2(q_k) dq_k dr \\
&= (h+p) \int_r g_1(r) \int_{q_k} q_k (r/q_k) g_3(rx_k/q_k) g_2(q_k) dq_k dr \\
&= (h+p) \int_r r g_1(r) \int_{q_k} g_3(rx_k/q_k) g_2(q_k) dq_k dr \qquad (4.9)
\end{aligned}$$

where $G_3(\cdot)$ is a cumulative distribution function of $g_3(\cdot)$. Note that the right hand side in (4.9) is nonnegative because all terms inside integrations are nonnegative. Thus, the expected cost function is convex and the solution satisfying the first-order condition provides an unique minimum. □

Toward this end, we now solve (4.8) under the assumption that the order size is uniformly distributed where $0 < q_k \leq N$, the random yield ranges uniformly between zero and one, and the unit processing time has a density of uniform, $U(0, T)$. Under the above assumptions, we state the following:

Proposition 4.2 *If $g_1(r) = 1$, $g_2(q_k) = 1/N$ and $g_3(\tau) = 1/T$, then the optimal solution to* (4.7) *is*

$$x_k^* = \frac{p}{(h+p)} E[Q_k](2T), \quad \textit{where} \quad E[Q_k] = \frac{N}{2}$$

Proof : Substituting $g_1(r), g_2(q_k)$ and $g_3(\tau)$ into (4.8) and solving for x_k gives

$$(h+p)\int_{r=0}^{1}\int_{q_k=0}^{N}\frac{rx_k^*}{NT}dq_kdr \quad - \quad pE[Q_k] \quad = \quad \frac{(h+p)x_k^*}{2T} - pE[Q_k] \quad = \quad 0$$

$$\Rightarrow x_k^* = \frac{p}{(h+p)}E[Q_k]2T \qquad \square$$

Therefore, in case that there is no order in the system when the kth order arrives at time t_k, the optimal solution, x_k^*, is a function of the maximum processing time, average order size as well as the holding cost and the tardiness cost. This solution along with $N = 1$ and $T = 1$ establishes its equivalence with a simple newsboy problem as Weeks (1981) showed. That is, in case that no orders are waiting or being processed in the system, our single stage problem is equivalent to a simple newsboy problem. Next, we consider the case that at least one order is in the system when a new order arrives at time t_k.

4.4.2 Case II: $t_k \leq s_{k-1}$

This is the case that when a new customer order arrives at time t_k, at least one preceded order is in the system. Referring to (4.6), we have

$$\begin{aligned}
\psi(x_k) &= E\left[hQ_k(t_k + x_k - s_k)^+\right] + E\left[pQ_k(s_k - t_k - x_k)^+\right] \\
&= h\int_{q_k} q_k g_2(q_k)\int_{s_k}(A_k - s_k)^+ g_4(s_k)ds_k dq_k \\
&\quad +p\int_{q_k} q_k g_2(q_k)\int_{s_k}(s_k - A_k)^+ g_4(s_k)ds_k dq_k
\end{aligned}$$

$$
\begin{aligned}
&= h\int_{q_k} q_k g_2(q_k) \int_{s_k=0}^{A_k} (A_k - s_k)\, g_4(s_k) ds_k dq_k \\
&\quad +p\int_{q_k} q_k g_2(q_k) \int_{s_k=A_k}^{\infty} (s_k - A_k)\, g_4(s_k) ds_k dq_k \qquad (4.10)
\end{aligned}
$$

where $A_k = t_k + x_k$ and g_4 is the density function of s_k.

We want to minimize (4.10) with respect to the planned customer leadtime, x_k. Again, assuming $\psi(x_k)$ is twice differentiable, we can prove the convexity of $\psi(x_k)$ as follows:

Proposition 4.3 *$\psi(x_k)$ is convex in $x_k > 0$*

Proof : From (4.10), it follows that

$$
\begin{aligned}
\frac{\partial \psi(x_k)}{\partial x_k} &= h\int_{q_k} q_k g_2(q_k) \frac{\partial}{\partial x_k}\left[\int_{s_k=0}^{A_k} (A_k - s_k)\, g_4(s_k) ds_k\right] dq_k \\
&\quad +p\int_{q_k} q_k g_2(q_k) \frac{\partial}{\partial x_k}\left[\int_{s_k=A_k}^{\infty} (s_k - A_k)\, g_4(s_k) ds_k\right] dq_k \\
&= h\int_{q_k} q_k g_2(q_k) \left[\int_{s_k=0}^{A_k} g_4(s_k) ds_k\right] dq_k \\
&\quad +p\int_{q_k} q_k g_2(q_k) \left[\int_{s_k=A_k}^{\infty} -g_4(s_k) ds_k\right] dq_k \\
&= \int_{q_k} q_k g_2(q_k) \left[h\int_{s_k=0}^{A_k} g_4(s_k) ds_k - p\int_{s_k=A_k}^{\infty} g_4(s_k) ds_k\right] dq_k \\
&= \int_{q_k} q_k g_2(q_k) \left[(h+p)\int_{s_k=0}^{A_k} g_4(s_k) ds_k - p\int_{s_k=0}^{\infty} g_4(s_k) ds_k\right] dq_k
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{q_k} q_k g_2(q_k) \left[(h+p)\int_{s_k=0}^{A_k} g_4(s_k) ds_k - p\right] dq_k \\
&= \int_{q_k} q_k \left[(h+p)G_4(A_k) - p\right] g_2(q_k) dq_k \qquad (4.11)
\end{aligned}
$$

where $G_4(\cdot)$ is a cumulative distribution function of s_k. Equating (4.11) to zero, the optimal solution x_k^* satisfies the following first order condition

$$(h+p)\int_{q_k} q_k G_4(t_k + x_k^*) g_2(q_k) dq_k - pE[Q_k] = 0 \tag{4.12}$$

and the second derivative is

$$\begin{aligned}\frac{\partial^2\psi(x_k)}{\partial {x_k}^2} &= (h+p)\int_{q_k} q_k g_2(q_k)\left[\frac{\partial}{\partial x_k}G_4(A_k)\right] dq_k \\ &= (h+p)\int_{q_k} q_k g_2(q_k) g_4(A_k) dq_k\end{aligned}$$

which is nonnegative since all terms inside the integrations are nonnegative. Thus, the expected cost function (4.10) is convex and the solution satisfying the first-order condition (4.12) provides the unique optimal. □

Unfortunately, the solution of (4.12) is, in many cases, not tractable analytically and we thus employ an numerical approximation to retrieve the solution.

4.5 Numerical Results

In this section, we study our model numerically to investigate the effect of changing values of cost parameters and average processing time on the optimal expected cost. For simplicity, we assume that each customer orders a single unit ($Q_k = 1$) and that there is no yield variability. We also assume that there are only two orders in the system when the kth order arrives at time t_k. The actual process completion time of order k is then, from (4.5),

$$\begin{aligned}s_k &= s_{k-2} + \tau_k + \tau'_{k-1} \\ &= s_{k-2} + \tau_k + \tau_{k-1} + U - t_k\end{aligned}$$

where U is the actual process starting time of order $(k-1)$. If the actual processing time of individual orders is exponentially distributed with a mean $1/\lambda$, the density function of s_k is

$$f(s_k) = \frac{(s_k - c)}{\lambda^2} e^{\frac{s_k - c}{\lambda}}, \quad c \leq s_k < \infty$$

$c = s_{k-2} + U - t_k$. Referring to (4.12), we get the following first-order condition:

$$(h+p) \int_c^{x_k + t_k} \frac{(s_k - c)}{\lambda^2} e^{\frac{s_k - c}{\lambda}} ds_k - p = 0$$

Since s_{k-2}, U and t_k are all known constants, we set $s_{k-2} = 5.0$, $U = 3.0$ and $t_k = 4.0$. The cost parameters considered for this problem are as follows: unit holding costs range $h = 1, 2, ..., 10$ and unit penalty costs are $p = 1, 2, ..., 10$. We also vary mean processing times, λ, between one time unit and fifty.

Numerical trials are conducted for a number of different parameter settings. For each parameter setting, we use simulation to estimate the optimal PLT, x_k^* and expected cost, $\psi(x_k^*)$, from (4.11). Figures 4.3 through 4.5 contain examples of our findings.

1. *Effect of average processing time on optimal PLT and expected cost.* Figure 4.3 presents the relationship between the average processing time and the optimal PLT, and between the average processing time and the optimal cost, given that the value of cost parameters are fixed. As we can see, both optimal PLT and expected cost increase with an increased processing time. This result is plausible because a longer

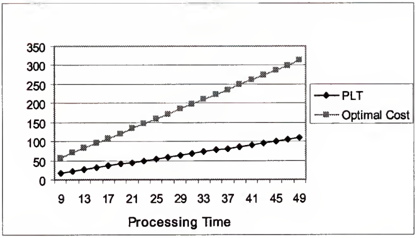
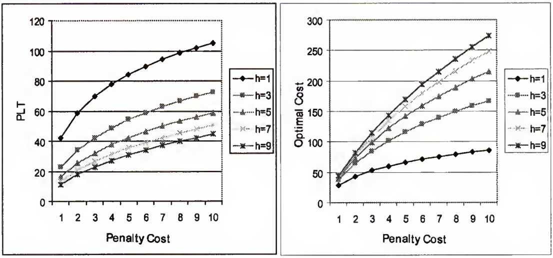


Figure 4.3: Optimal cost and PLT for different mean processing time.

processing time causes a larger PLT and thus more holding cost.

2. *Increasing penalty cost.* With increased penalty cost, the optimal PLT increases for any levels of holding cost, as in Figure 4.4 (a), assuming that the average processing time remains a constant. It is obvious that when penalty costs are high, it is inclined to have a larger PLT to avoid a high penalty cost. Furthermore, we also notice that the optimal expected cost is proportional to the penalty cost (see Figure 4.4 (b)). In other words, high tardiness costs cause high optimal expected costs. This is true because, given that any other parameters are unchanged, high penalty or holding costs result in high overall costs.

3. *Increasing holding cost.* In Figure 4.5 (a), note that as the holding cost is increased, the optimal PLT decreases for any levels of penalty cost. One explanation is the fact that, since a holding cost occurs when the PLT of an order exceeds the



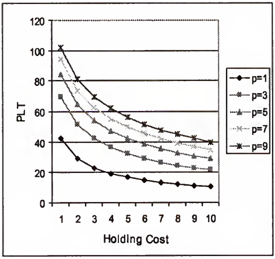
(a) (b)

Figure 4.4: Optimal PLT and overall cost for different penalty cost. (a) PLT vs penalty cost. (b) Overall cost vs penalty cost. $\lambda = 27$.

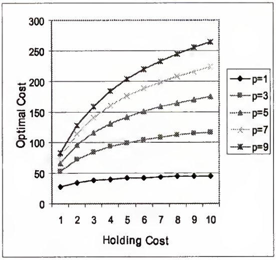
actual job-completion time, the optimal PLT is forced to be decreased to avoid a high holding cost, given that other parameters are remained unchanged. On the other hands, as we can see in Figure 4.5 (b), the optimal cost increases as the holding cost increases, given that other parameters are remained the same.

4.6 Conclusions

Stochastic production systems with limited resources are discussed. We have modeled the problem of setting optimal planned order leadtime in a single stage, make-to-order production system. As a solution approach, we derived the distribution of actual process completion time of individual orders and then determined the optimal planned leadtime. We also identify that our model is equivalent to a simple newsboy problem when there is only one customer order (the newly arrived



(a)



(b)

Figure 4.5: Optimal PLT and overall cost for different holding cost. (a) PLT vs holding cost. (b) Overall cost vs holding cost. $\lambda = 27$.

one itself) in the system. For the case that multiple orders reside in the system, we have shown, through the computational studies, that the optimal PLT is proportional to a unit penalty cost, but is reciprocal to a unit holding cost, whereas the optimal expected cost for our lead-time model is proportional to both penalty and holding costs. These results are rather acceptable because when a tardiness cost is much higher than a holding cost, manufacturers try to offer longer PLT's to their customers to avoid expensive penalty costs, and vice versa.

CHAPTER 5
CONCLUSIONS AND FUTURE RESEARCH

This dissertation investigates issues on stochastic systems with limited resources. We study stochastic multi-item inventory models with limited warehouse capacity for the cases of both equal and unequal replenishment intervals. We also discuss stochastic production systems in which a single machine serves a series of customer orders. The primary contributions of this dissertation are

- The development of heuristics to estimate replenishment quantities when a warehouse confronts a limited capacity and individual products have unequal replenishment intervals.
- The derivation of the optimal order leadtime in a single-stage, make-to-order production system.

We first characterize a stochastic multi-item inventory model for the case of equal replenishment intervals with limited warehouse capacity by using dynamic programming. We use induction to derive the convexity of objective cost function and show that a myopic replenishment policy is optimal for both a finite and an infinite planning horizon. Furthermore, the optimal policy in each period is easy to calculate due to the independence of the one period expected cost over periods.

For the unequal replenishment interval case, we completely characterized the optimal replenishment policy for the two-item, two-period case. For more general cases, because of the inability to show convexity of the objective function, we construct three heuristics to determine replenishment quantities for the case of unequal replenishment intervals.

Numerical testing of these heuristics suggests that they yield near optimal solutions for a small set of problems. The performance of these heuristics has been evaluated for a number of problems with various cost and demand parameters. When there is enough slack in the resource constraint, all heuristics demonstrate identical results.

The first heuristic, denoted Heuristic A, which uses a separate capacity for each product for replenishment, is simple enough for practical use, but is not able to detect a scarce resource well, compared to the other two heuristics, in most problems. This behavior is plausible because the heuristic is more likely to retain unused capacities for some products while some others suffer from the lack of the resource.

Heuristic C adopts a replenishment policy which replenishes products up to the total available capacity of the system at the time instant of replenishment. This heuristic outperforms Heuristic A in most cases because it does not carry the capacity-inefficiency confronted by the Heuristic A , but leads to poorer performance for the problems with high penalty costs. This is because, in this situation, Heuristic C is too aggressive to replenish products in one period so that the warehouse suffers from the lack of resource in the following replenishment period, causing high penalty costs to the warehouse.

On the other hand, the second heuristic, denoted Heuristic B, adjusts separate capacities in each replenishment period, based on the availability of any unused resource. This characteristic discriminates Heuristic B from Heuristic A and diminishes the capacity-inefficiency of Heuristic A. Furthermore, the capacity-separation scheme of the heuristic leads to better performance than Heuristic C.

Stochastic production systems with limited resources are also discussed. We have modeled the problem of setting optimal planned order leadtime in a single stage, make-to-order production system. As a solution approach, we derived the distribution of actual process completion time of individual orders and then determined the optimal planned leadtime. We also identify that our model is equivalent to a simple newsboy problem when there is only one customer order (the newly arrived one itself) in the system. For the case that multiple orders reside in the system, we have shown, through the computational studies, that the optimal PLT is proportional to a unit penalty cost, but is reciprocal to a unit holding cost, whereas the optimal expected cost for our lead-time model is proportional to both penalty and holding costs. These results are rather acceptable because when a tardiness cost is much higher than a holding cost, manufacturers try to offer longer PLT's to their customers to avoid expensive penalty costs, and vice versa.

We conclude this dissertation by pointing out possible extensions of the stochastic models considered in this study. The current optimization model is established for the fixed warehouse capacity over time. One interesting development is to consider

a variable warehouse capacity. This extension is important, since in many contexts the warehouse capacity would be variable. For example, extra warehouse(s) can be leased to accommodate more inventories to satisfy customer demands. Or Federal Express uses its giant warehouse for product replenishment as well as for its daily operation, such as mail classification. The total available warehouse capacity then varies according to the size and/or the duration of daily operation. In this case, the warehouse capacity in period j, say V_j, varies over periods and we redefine the problem in (3.17) to determine the separate capacity of each item for Heuristic A as following

$$\begin{aligned} Find \quad & \mu_j^* = \min\{\mu_j \geq 0\} \\ s.t. \quad & \sum_{i=1}^{l} V_{i,j} \leq V_j \\ & V_{i,j} = F_i^{-1}\left(\frac{b_i p_i - \mu_j}{b_i(h_i + p_i)}\right), \quad i = 1, 2, ..., l \end{aligned}$$

where $V_{i,j}$ is the separate capacity of item i in period j. We also can redefine the processes for Heuristics B and C using the same manner as above to determine the replenishment policies.

Another research effort may continue along by relaxing the assumption of zero delivery leadtimes between manufacturers and a warehouse. It is also of interest to analyze the warehousing problem with the additional assumption of nonzero fixed ordering costs.

APPENDIX A

There are two cases to solve equations (b)-(d) and (4) in the single-item case. Assuming that F^{-1} exists,

(i) : $I + Q - V_0 \neq 0$.

In this case, we get $\mu = 0$ from (c), and then the following is immediate from (4):

$$Q = F^{-1}\left(\frac{p}{h+p}\right) - I \tag{A.1}$$

Combining (b) and the condition (i), we have:

$$\begin{aligned}
& I + Q - V_0 \leq 0 \\
& or \;\; F^{-1}\left(\frac{p}{h+p}\right) - V_0 < 0 \quad \text{from } (A.1) \\
\Rightarrow \quad & Q = F^{-1}\left(\frac{p}{h+p}\right) - I \qquad if \;\; F^{-1}\left(\frac{p}{h+p}\right) < V_0
\end{aligned}$$

(ii) : $I + Q - V_0 = 0$. In this case, it is obvious that $Q = V_0 - I$ $(*)$ from the condition (ii). Then, from (2.4) and (c),

$$\begin{aligned}
& \mu = p - (h+p)F(I+Q) \geq 0 \\
& or \;\; I + Q \leq F^{-1}\left(\frac{p}{h+p}\right) \\
& or \;\; V_0 \leq F^{-1}\left(\frac{p}{h+p}\right) \quad \text{from } (*)
\end{aligned}$$

$$\Rightarrow \quad Q = V_0 - I \qquad\qquad if \quad V_0 \leq F^{-1}\left(\frac{p}{h+p}\right)$$

Combining those two results, we have:

$$Q^* = \begin{cases} F^{-1}\left(\frac{p}{h+p}\right) - I & if \quad F^{-1}\left(\frac{p}{h+p}\right) < V_0 \\ V_0 - I & if \quad F^{-1}\left(\frac{p}{h+p}\right) \geq V_0 \end{cases}$$

Or simply

$$Q^* = min\left[F^{-1}\left(\frac{p}{h+p}\right), V_0\right] - I$$

APPENDIX B

The first and second partial derivatives of g_j with respect to $Q_{i,j}$ can be determined as follows:

$$
\begin{aligned}
g_j(I_{1,j},...,I_{n,j},Q_j) &= g_j(I_{1,j},...,I_{n,j},Q_{1,j},...,Q_{n,j}) \\
&= \sum_{i=1}^{l} E\left[h_i(I_{i,j}+Q_{i,j}-D_{i,j})^+ + p_i(D_{i,j}-I_{i,j}-Q_{i,j})^+\right] \\
\frac{\partial}{\partial Q_{i,j}} g_j &= \frac{\partial}{\partial Q_{i,j}} \sum_{i=1}^{l} E\left[h_i(I_{i,j}+Q_{i,j}-D_{i,j})^+\right. \\
&\qquad \left. + p_i(D_{i,j}-I_{i,j}-Q_{i,j})^+\right] \\
&= \frac{\partial}{\partial Q_{i,j}} \sum_{i=1}^{l} \left[\int_0^{A_{i,j}} h_i(I_{i,j}+Q_{i,j}-x) f_{i,j}(x)dx\right. \\
&\qquad \left. + \int_{A_{i,j}}^{\infty} p_i(x-I_{i,j}-Q_{i,j}) f_{i,j}(x)dx\right] \\
&= \int_0^{A_{i,j}} h_i f_{i,j}(x)dx + \int_{A_{i,j}}^{\infty} p_i f_{i,j}(x)dx \\
&= (h_i+p_i)\,F_{i,j}\,(I_{i,j}+Q_{i,j}) - p_i \\
\frac{\partial^2}{\partial {Q_{i,j}}^2} g_j &= (h_i+p_i)\,f_{i,j}\,(I_{i,j}+Q_{i,j}) \qquad \text{(B.1)} \\
\frac{\partial^2}{\partial Q_{i,j}Q_{i',j}} g_j &= \frac{\partial^2}{\partial Q_{i',j}} \left[(h_i+p_i)\,F_{i,j}\,(I_{i,j}+Q_{i,j}) - p_i\right] \\
&= 0 \qquad \text{(B.2)}
\end{aligned}
$$

where $A_{i,j} = I_{i,j} + Q_{i,j}$ and $F_{i,j}$ is the cumulative distribution function of demand $D_{i,j}$. Since the right-hand side of (B.1)is always nonnegative and (B.2) is equal to zero, the Hessian matrix B_j is positive semi-definite.

APPENDIX C

We use the induction method. First, consider the period m. Define $L_m = G_m(I_{1,m},...,I_{l,m},Q_{1,m},...,Q_{l,m}) + \mu_m\left(\sum_{i=1}^{l} I_{i,m} + \sum_{i=1}^{l} Q_{i,m} - V_0\right)$. Then, from the first-order condition, we get

$$
\begin{aligned}
\frac{\partial L_m}{\partial Q_{i,m}} &= \frac{\partial}{\partial Q_{i,m}} G_m + \frac{\partial}{\partial Q_{i,m}}\left[\mu_m(\sum_{i=1}^{l} I_{i,m} + \sum_{i=1}^{l} Q_{i,m} - V_0)\right] \\
&= \frac{\partial}{\partial Q_{i,m}} \sum_{i=1}^{l}\left[h_i \int_0^{I_{i,m}+Q_{i,m}} (I_{i,m}+Q_{i,m}-x_i) f_i(x_i) dx_i \right. \\
&\qquad \left. + p_i \int_{I_{i,m}+Q_{i,m}}^{\infty} (x_i - I_{i,m} - Q_{i,m}) f_i(x_i) dx_i \right] \\
&\quad + \frac{\partial}{\partial Q_{i,m}}\left[\mu_m(\sum_{i=1}^{l} I_{i,m} + \sum_{i=1}^{l} Q_{i,m} - V_0)\right] \\
&= h_i \int_0^{I_{i,m}+Q_{i,m}} f_i(x_i) dx_i - p_i \int_{I_{i,m}+Q_{i,m}}^{\infty} f_i(x_i) dx_i + \mu_m \\
&= h_i F_i(I_{i,m}+Q_{i,m}) - p_i(1 - F_i(I_{i,m}+Q_{i,m})) + \mu_m \\
&= (h_i + p_i) F_i(I_{i,m}+Q_{i,m}) - p_i + \mu_m = 0 \qquad \text{(C.1)}
\end{aligned}
$$

Solving (C.1) and the KKT condition: $\mu_m(\sum_{i=1}^{l} I_{i,m} + \sum_{i=1}^{l} Q_{i,m} - V_0) = 0$, we get

$$
\begin{aligned}
Q_{i,m}^* &= F_i^{-1}\left(\frac{p_i + \mu_m^*}{h_i + p_i}\right) - I_{i,m} \\
\mu_m^* &= arg \min_{\mu_m \geq 0}\left[\sum_{i=1}^{l} F_i^{-1}\left(\frac{p_i - \mu_m}{h_i + p_i}\right) \leq V_0\right]
\end{aligned}
$$

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